Theory and Modeling of the Radiation Belts

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Theory and Modeling of the Radiation Belts

Contents

OUTLINE:

Introduction

- **Aim:**

  To review current theory and modeling of the radiation belts (with emphasis on relativistic electrons during magnetic storms)

- **Radiation belts:** trapped particles $\gtrsim$ 500 keV

- **Relativistic electrons** are of special interest because high fluxes of these particles:
  - Are associated with spacecraft operational anomalies.
  - Are hazardous to humans in space.
  - Affect the middle atmosphere through precipitation.

- **The basic physical processes responsible for these high fluxes are not well understood.**

Observations

- **Radiation belts vary on three main time scales:**

  1. Quasi-static variations on time scales of months

  2. Variations of MeV electron flux over hours–days

  3. Strong flux increases on time scales of minutes

1. Quasi-static variations on time scales of months

- Quiet-times, between storms
- E.g., several-month decay of March '91 fluxes
- Extensive body of early work [*Schulz and Lanzerotti*, 1974]
- Assume radial and pitch-angle diffusion with static coefficients

2. Variations of MeV electron flux over hours–days

- Associated with high speed stream, CME, and magnetic cloud storms
  [E.g., *Baker et al.*, 1986, 1997, 1998; *Reeves et al.*, 1998]
- Often show similarity between log(flux) and *Dst*
  Flux decrease followed by large increase
- Flux decreases reproduced by models
  [E.g., *Kim and Chan*, 1997]
- Physical mechanism for the flux increases?

3. Strong flux increases on time scales of minutes

- Associated with sudden compression of the magnetosphere
- Strong flux enhancements on particle-drift time scales
- Examples: March 24, 1991 and August 18, 1991
- Highly successful modeling of these events
  [*Li et al.*, 1993; *Hudson et al.*, 1995]

Overview of Physical Mechanisms

Mechanisms for the Flux Decreases

- **Loss to the Atmosphere**
  - Pitch-angle scattering into the atmospheric loss cone.
  - Precipitating electrons are observed.

    But not enough to account for observed flux decreases?

- **Loss to the Magnetopause**
  - Changes in the global magnetospheric configuration results in particles drifting into the magnetopause.
  - The dayside magnetopause can be pushed inside 6.6 R_E

- **Fully-Adiabatic Flux Changes**
  - All three adiabatic invariants are conserved.
  - Flux changes result from mapping of the constant phase-space density.

What Causes the Flux Increases?

- **Entry of energetic electrons from outside the magnetosphere?**

  E.g., electrons associated with solar flares, CIR shocks, Jovian electrons

- **Probably not the dominant source:**
  - For some events the fluxes rise first deep inside the magnetosphere.
  - Solar wind phase-space densities are too low to supply observed magnetospheric fluxes.
    [*Li et al.*, 1997]

- Thus, **internal** acceleration mechanisms seem more important...

Proposed mechanisms for the flux increases:

- Recirculation
  [*Nishida*, 1976; *Fujimoto and Nishida*, 1990]

- Cyclotron-resonant heating by whistler-mode waves
  [*Temerin et al.*, 1994; *Summers et al.*, 1998]

- Fully-adiabatic flux changes (recovery phase)
  [E.g., *Kim and Chan*, 1997]

- Substorm injection of energetic plasma sheet electrons
  [E.g., *Chan et al.*, 1997; *Kim*, 1999]

- Diffusion of trapped energetic electrons from the cusp
  [*Sheldon et al.*, 1998]

- Drift-resonant acceleration by MHD waves
  [*Hudson et al.*, 1998; *Chan and Hudson*, 1998]

- Acceleration and pitch angle scattering by ULF and whistlers
  [*Liu et al*, 1999]

1. Theoretical Foundations

Adiabatic Theory

Useful basic references: *Northrop and Teller* [1960]; *Kruskal* [1962]; *Northrop* [1963]; *Roederer* [1970]; *Schulz and Lanzerotti* [1974]; *Wolf* [1983]; *Schulz* [1991]

- A charged particle *trapped* in a magnetic field $\mathbf{B}$ undergoes motion with three distinct frequencies:
  - Cyclotron: $\omega_c = eB/mc$
  - Bounce: $\omega_b \approx v_{\|}/L$ (where L is the field line length)
  - Drift: $\omega_d \approx v_d/r_d$ (where v_d is the ∇B and curvature drift velocity and r_d is the radius of the drift orbit)

  Each of these is associated with a corresponding **phase angle**: gyrophase, bounce phase, and drift phase.

- In **guiding center (GC) theory**: use the separation of the gyroperiod from other time scales in the system to define a small parameter ϵ_0, where

  $$\epsilon_0 \sim |\partial \ln B/\partial t|/\omega_c \sim \rho|\nabla \ln B| \ll 1.$$

  This **GC parameter** ϵ_0 is used to remove the gyrophase dependence from the system, order by order in ϵ_0

- **To lowest order:** average the eqs of motion over the gyrophase angle [*Alfvén*, 1950; *Northrop*, 1963]

- Removal of gyrophase dependence **to higher orders** in ϵ_0 is more difficult ...

  Gyrophase dependence can be removed to **arbitrary order** using **phase-space Lagrangian Lie transform** methods [*Littlejohn*, 1982; *Brizard*, 1989; *Chan*, 1991]

- Removal of gyrophase dependence to a given order in ϵ_0 implies conservation of a conjugate quantity to that order in ϵ_0

  The conserved quantity, called the **first adiabatic invariant**, μ, is an asymptotic series in powers of ϵ_0

- To zeroth order in ϵ_0 $\boxed{\mu = \mu_0 = p_\perp^2/2mB}$
  $\mathbf{p} = m\gamma\mathbf{v}$ is the relativistic momentum and B is evaluated at the GC position

- **Advantages of GC equations:**

  The phase space is **reduced** from 6D to 4D

  Removal of fast gyromotion $\Rightarrow$ **much larger time steps** in computer simulations

Bounce motion and the second invariant

- The guiding center system can be **further reduced:**
  - Define a second small parameter ϵ_b=(bounce period)/(next-higher time scale)
  - Remove the bounce-phase dependence order by order in ϵ_b
  - Obtain the **second invariant** J

  To lowest-order

  $$J = \oint p_{\|} \, ds$$

  and the equations of motion are **bounce-averaged**

  [*Northrop and Teller*, 1960; *Wolf*, 1983]

- A better invariant than J is the energy-independent quantity

  $$\boxed{K = \frac{J}{\sqrt{8m\mu}} = \int \sqrt{B_m - B} \, ds}$$

  where B_m is the mirror point field.

  K is a function of configuration space coordinates only

Drift motion and the third invariant

- Finally, if $\omega_d \gg$ (any remaining frequency)
  the **third invariant** is Ψ, the magnetic flux linked by the "bounce center" during a drift orbit.

- An alternative to Ψ is the "Roederer L shell":

$$\boxed{L^* = \frac{2\pi \mathcal{M}_E}{R_E \Psi}}$$

  where $\mathcal{M}_E$ is the planetary magnetic moment.

- (μ, K, L^*) are a preferred set of coordinates for radiation belt studies

  [*Schulz*, 1996; *Selesnick et al.*, 1997; *Selesnick and Blake*, 1998]

Radiation Belt Transport Equations

- An adiabatic invariant can be broken when there is a change on a time scale $\lesssim$ the time scale of the corresponding periodic motion.

- Processes which break adiabatic invariants may lead to transport described by a Fokker-Planck equation:

$$\frac{\partial \overline{f}}{\partial t} = \sum_{i,j=1}^{3} \frac{\partial}{\partial J_i}\left(D_{ij}\frac{\partial \overline{f}}{\partial J_j}\right) - \sum_{i=1}^{3} \frac{\partial}{\partial J_i}\left(\Gamma_i \overline{f}\right) + S - \frac{\overline{f}}{\tau_L}$$

  Here

$\overline{f}$	is the phase-space density averaged over phase angles
J_i	are the three adiabatic invariants
D_{ij}	are diffusion tensor elements
Γ_i	are drag coefficients
S, τ_L	represent sources and losses

- Transport equations in coordinates other than (J_1, J_2, J_3) (such as (μ, K, L^*)) are obtained by inserting Jacobian factors

2. Modeling Methods

- **Important features of radiation belt particles:**
  - High energy, low density, trapped particles
  - Magnetic drifts dominate convection $\mathbf{E} \times \mathbf{B}$ drifts
  - Do not significantly affect electromagnetic fields

- **Thus:**
  - Test-particle calculations are accurate
  - Theoretical models are relatively simple (conceptually)
  - Simulation codes are relatively fast (highly parallelizable)

- **Two main approaches to radiation belt modeling:**
  - The "Liouville approach"
  - The "Fokker-Planck approach"

The Liouville Approach

- Compute test-particle trajectories and apply Liouville's theorem

- Lorentz-force equations: straightforward but expensive
  Use time-reversed orbits and reduced dynamics
  E.g., guiding-center or bounce-averaged

- Requires:
  - **B** and **E** fields on length and time scales of interest
  - Test-particle initial and boundary conditions
  - Initial and boundary phase-space densities

- Examples:
  - Simulation of the March 1991 event [*Li et al.*, 1993; *Hudson et al.*, 1995]
  - Simulation of the January 1997 event [*Hudson et al.*, 1998]
  - Response of MeV electrons to storm-time **B** changes [*Kim and Chan*, 1997]

The Fokker-Planck Approach

- Solve a Fokker-Planck equation for a phase-averaged phase-space density $\overline{f}$

- Early radial diffusion calculations of this type.

  Recent work exemplified by the Salammbô code

  [*Beutier et al.*, 1995; *Bourdarie et al.*, 1997]

- Processes which break the 3 adiabatic invariants represented by transport coefficients such as D_{LL}, D_{LJ}, D_{JJ}, $D_{J\mu}$, etc.

- Requires
  - Initial and boundary conditions for $\overline{f}$
  - Magnetospheric magnetic field model
  - Transport coefficients, loss lifetimes, source terms, ...

- Enables inclusion of high-frequency wave-particle interactions ($\sim\omega_{ce}$) through $D_{\mu\mu}$, etc.

  [E.g., *Temerin et al.*, 1994; *Horne and Thorne*, 1998]

Comparison of Liouville and Fokker-Planck Approaches

- Liouville approach contains more complete physics, but this is probably not always necessary

- Fokker-Planck approach can give large savings in computer time, but underlying assumptions may not be justified

- Both approaches should be developed...
  - Model test cases and real events
  - Compare with observations and with one another
  - Multisatellite data sets will be crucial

A Comment on Data Analysis:

- Plotting phase-space density as a function of (μ, K, L^*) is very useful for separating adiabatic and nonadiabatic behavior, including
  - Sources
  - Losses
  - Transport and acceleration

- This requires:
  - Large range and good resolution in r, energy and pitch angle
  - A reliable magnetic field model

  [E.g., *Selesnick et al.*, 1997; *Selesnick and Blake*, 1998]

- Multisatellite plots of $\overline{f}(\mu, K, L^*)$ are especially interesting

3. Theoretical Foundations Revisited

Phase-Space Lagrangian Methods

Advantages of the PSL methods:

1. A Hamiltonian formulation allows **a general picture of resonant wave-particle interactions** in terms of breaking of adiabatic invariants

2. **Hamiltonian conservation properties** (e.g., conservation of energy and phase-space volume) **are retained**

3. **Not restricted** to using canonical variables.

4. Lie transforms provide a **systematic efficient method** to calculate high-order GC equations.

5. The methods allow **many extensions**:
   - A conserved quantity for time-dependent systems
   - $\omega \ll \omega_c$ wave-particle interactions [*Chan et al.*, 1989]
   - Relativistic motion [E.g., *Brizard and Chan*, 1999]
   - Drift-kinetic and gyrokinetic equations [*Brizard*, 1989]

The Phase-Space Lagrangian

- **Traditionally,** Hamiltonian mechanics is defined by
  - a scalar function H (the Hamiltonian), and
  - a special set of phase-space coordinates $(\mathbf{q}, \mathbf{p})$ (the canonical coordinates)

  such that the time evolution of the system is given by

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

  (Hamilton's equations)

  In general, $\mathbf{q}$ and $\mathbf{p}$ are N–vectors and H is a function of $(\mathbf{q}, \mathbf{p}, t)$

- **Alternatively,** Hamiltonian mechanics can be derived from the variational principle

$$\delta \int L(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}, t)\, dt = 0$$

  where the function

$$\boxed{L(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}, t) \equiv \mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)}$$

  is called the **phase-space Lagrangian (PSL)**

- These two approaches are equivalent,
  but **the PSL is not restricted to canonical coordinates**

The equations of motion

- Consider an arbitrary phase-space coordinate transform given by the $2N$ functions $z^i = z^i(\mathbf{q}, \mathbf{p}, t) \quad \text{for} \quad i = 1, \ldots, 2N$

- By the chain rule

$$\boxed{L(\mathbf{z}, \dot{\mathbf{z}}, t) = \gamma_i(\mathbf{z}, t)\dot{z}^i - h(\mathbf{z}, t)}$$

  where

$$\gamma_i(\mathbf{z}, t) = \mathbf{p} \cdot \frac{\partial \mathbf{q}}{\partial z^i} \qquad h(\mathbf{z}, t) = H(\mathbf{q}, \mathbf{p}, t) - \mathbf{p} \cdot \frac{\partial \mathbf{q}}{\partial t}$$

  L is the PSL in **arbitrary** phase-space coordinates.

  It is **the central object of noncanonical Hamiltonian mechanics**

- In these coordinates the variational principle yields the following **Euler-Lagrange equations**:

$$\omega_{ij}\dot{z}^j = \partial_i h + \partial_t \gamma_i$$

  where $\partial_i \equiv \partial/\partial z^i$ and

$$\omega_{ij} = \partial_i \gamma_j - \partial_j \gamma_i$$

The equations of motion ...

- It can be shown that:
  - ω_{ij} is a **matrix of Lagrange brackets**

$$\omega_{ij} = [z^i, z^j] = \partial_i \mathbf{p} \cdot \partial_j \mathbf{q} - \partial_j \mathbf{p} \cdot \partial_i \mathbf{q}$$

  - The inverse of ω_{ij} is a **matrix of Poisson brackets**

$$\omega_{ij}^{-1} = \{z^i, z^j\} = \partial_{\mathbf{q}} z^i \cdot \partial_{\mathbf{p}} z^j - \partial_{\mathbf{p}} z^i \cdot \partial_{\mathbf{q}} z^j$$

- Using $J^{ij} \equiv \omega_{ij}^{-1}$ the Euler-Lagrange equations can be solved explicitly for the time derivatives:

$$\dot{z}^i = J^{ij}(\partial_j h - \partial_t \gamma_j)$$

Hamilton's equations in arbitrary phase-space coordinates

- The matrix ω_{ij}^{-1} can be used to find $\{f, H\}$.

  Then we have

$$\boxed{\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}}$$

A fundamental example: charged particle motion

1. The **canonical Hamiltonian** for nonrelativistic motion is

$$H(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q}, t)\right]^2 + e\Phi(\mathbf{q}, t)$$

where

$$\mathbf{q} = \mathbf{x} \qquad \text{and} \qquad \mathbf{p} = m\mathbf{v} + (e/m)\mathbf{A}(\mathbf{q}, t)$$

($\mathbf{x}$ and $\mathbf{v}$ are the particle position and velocity)

2. The corresponding **phase-space Lagrangian** is

$$L = \mathbf{p} \cdot \dot{\mathbf{q}} - \left\{\frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q}, t)\right]^2 + e\Phi(\mathbf{q}, t)\right\}$$

3. In the noncanonical **but more physical** phase-space coordinates $\mathbf{z} = (\mathbf{x}, \mathbf{v})$ the PSL becomes

$$\boxed{L = \left(\frac{e}{c}\mathbf{A} + m\mathbf{v}\right) \cdot \dot{\mathbf{x}} - \left(\frac{1}{2}mv^2 + e\Phi\right)}$$

4. The corresponding **Euler-Lagrange equations** are

$$\dot{\mathbf{x}} = \mathbf{v} \qquad \text{and} \qquad \dot{\mathbf{v}} = \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

The usual Newton-Lorentz equations of motion

Application to Guiding Center Theory

The Guiding Center Phase-Space Lagrangian

- **Starting point:** the PSL for motion in a static electromagnetic field $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\nabla \Phi$ is

$$\gamma = \left(\frac{e}{c} \mathbf{A} + m\mathbf{v} \right) \cdot d\mathbf{x} - \left(\frac{1}{2} m v^2 + e\Phi \right) \, dt$$

- **In dimensionless form:**

$$\gamma = \frac{1}{\epsilon_0} \left(\mathbf{A} + \epsilon_0 \mathbf{v} \right) \cdot d\mathbf{x} - \left(\frac{1}{2} v^2 + \Phi \right) \, dt$$

where $\epsilon_0 \equiv \rho_0 / R_0 \ll 1$, with $\rho_0 = v_0 / \omega_{c0}$, the characteristic gyroradius.

- **Transforming** from $(\mathbf{x}, \mathbf{v})$ to $(\mathbf{x}, v_\parallel, \mu_0, \xi)$ where

  $v_\parallel$ is the parallel velocity

  $\mu_0 \equiv v_\perp^2 / (2B)$

  ξ is the gyrophase

  the PSL transforms to

$$\gamma' = \frac{1}{\epsilon_0} \left[\mathbf{A} + \epsilon_0 (v_\parallel \widehat{\mathbf{b}} + \rho B \hat{\mathbf{c}}) \right] \cdot d\mathbf{x} - \left(\frac{1}{2} v_\parallel^2 + \mu_0 B + \Phi \right) \, dt$$

The Guiding Center Phase-Space Lagrangian...

- **To first order in** ϵ_0 the guiding center PSL is

$$\begin{aligned} \Gamma &= \left(\frac{1}{\epsilon_0}\mathbf{A} + U\widehat{\mathbf{b}}\right) \cdot d\mathbf{X} + \epsilon_0 \mu \, d\Xi - H \, dt \\ H &= \frac{1}{2}U^2 + \mu B + \Phi + \mathcal{O}(\epsilon_0^2) \end{aligned}$$

All functions are evaluated at the GC position $\mathbf{X}$

- The GC phase-space coordinates $(\mathbf{X}, U, \mu, \Xi)$ are related to the particle coordinates $(\mathbf{X}, v_\parallel, \mu_0, \xi)$ by

$$\begin{aligned} \mathbf{X} &= \mathbf{x} - \epsilon_0 \boldsymbol{\rho} + \mathcal{O}(\epsilon_0^2) \\ U &= v_\parallel + \mathcal{O}(\epsilon_0) \\ \mu &= \mu_0 + \mathcal{O}(\epsilon_0) \\ \Xi &= \xi + \mathcal{O}(\epsilon_0) \end{aligned}$$

Hamiltonian Guiding Center Equations of Motion

- **To first order in** ϵ_0 the Euler-Lagrange equations are:

$$\boxed{\dot{\mu} = 0}$$

showing that μ is an exact constant of the motion (to this order),

$$\dot{\Xi} = \frac{1}{\epsilon_0} B$$

showing the fast gyromotion,

$$U = \widehat{\mathbf{b}} \cdot \dot{\mathbf{X}}$$

showing that U is indeed the GC parallel velocity,

$$\boxed{\dot{\mathbf{X}} = \frac{1}{B_\parallel^*}\left[U\mathbf{B}^* + \epsilon_0 \widehat{\mathbf{b}} \times (\mu \nabla B - \mathbf{E})\right]}$$

which contains the curvature, ∇B, and $\mathbf{E} \times \mathbf{B}$ drifts, and

$$\boxed{\dot{U} = \frac{\mathbf{B}^*}{B_\parallel^*} \cdot (\mathbf{E} - \mu \nabla B)}$$

showing the $E_\parallel$ acceleration and the mirror force

(Here $\mathbf{B}^* \equiv \mathbf{B} + \epsilon_0 U \nabla \times \widehat{\mathbf{b}}$ and $B_\parallel^* = \widehat{\mathbf{b}} \cdot \mathbf{B}^*$)

- To order ϵ_0 these are the GC equations of *Northrop* [1963]

- Second order terms (such as those in the factor $\mathbf{B}^*/B_\parallel^*$) ensure **the Hamiltonian properties are preserved**

The Relativistic Guiding Center PSL

- The **nonrelativistic** GC PSL

$$L = \left(\frac{q}{c}\mathbf{A} + m\mathbf{u}_\parallel\right) \cdot \dot{\mathbf{X}} - H$$

with Hamiltonian

$$H = \frac{1}{2}mu_\parallel^2 + \mu B + q\Phi$$

is easily generalized...

- Use the relativistic momentum $\mathbf{p} = m\gamma\mathbf{v}$
  $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor

- The relativistic guiding center PSL is

$$\boxed{L = \left(\frac{q}{c}\mathbf{A} + \mathbf{p}_\parallel\right) \cdot \dot{\mathbf{X}} + \epsilon_0\mu\dot{\Theta} - H}$$

with Hamiltonian

$$\boxed{H = \sqrt{p_\parallel^2 c^2 + 2mc^2\mu B + m^2c^4} + q\Phi}$$

and $\mu = p_\perp^2/2mB$ is the relativistic first adiabatic invariant

Resonance Islands and Resonance Overlap

A Generic Picture of Wave-Particle Interactions

- Consider a periodic system subject to a perturbation

  E.g., motion of a trapped particle subject to waves

- The unperturbed motion is very simple when expressed in action-angle variables $(\mathbf{J}, \boldsymbol{\theta})$

$$\dot{\mathbf{J}} = \mathbf{0} \qquad \dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}(\mathbf{J})$$

- The equations of motion in terms of the action variables may be written in the form

$$\dot{\mathbf{J}} = \sum_{\mathbf{n}} \mathbf{A}_{\mathbf{n}}(\mathbf{J}, t) e^{i\mathbf{n}\cdot\boldsymbol{\theta}}$$

  where $\mathbf{n}$ is a vector index of integers and the coefficients $\mathbf{A}_{\mathbf{n}}(\mathbf{J}, t)$ are Fourier coefficients of the perturbed velocities

- $\mathbf{A}_{\mathbf{n}}(\mathbf{J}, t)$ are called the wave-particle coupling coefficients

A Generic Picture of Wave-Particle Interactions ...

- Integrating along the unperturbed orbits yields the resonance condition

$$\omega - \mathbf{n} \cdot \mathbf{\Omega}(\mathbf{J}) = 0,$$

- For small perturbations, the trajectories in the neighborhood of the resonant surfaces form island chains

- For larger perturbation amplitude the islands may "overlap", resulting in large-scale chaotic motion

  [*Chirikov*, 1979]

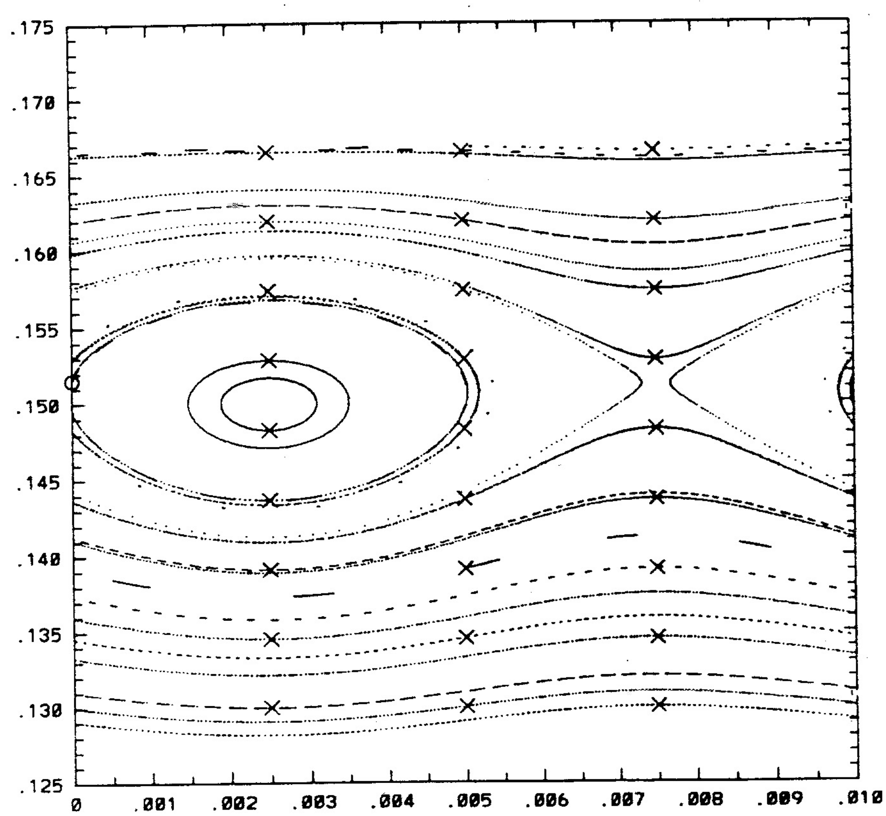
- If the islands overlap sufficiently, the motion may be approximated by a quasilinear diffusion tensor, D_{ij}

  D_{ij} are proportional to the square of the wave-particle coupling coefficients.

  [*Kaufman*, 1972]

Motion in One Mode

- **Poincaré plot:** J_ψ vs θ_ψ when $\theta_\chi = 0 \; (\text{mod} 2\pi)$

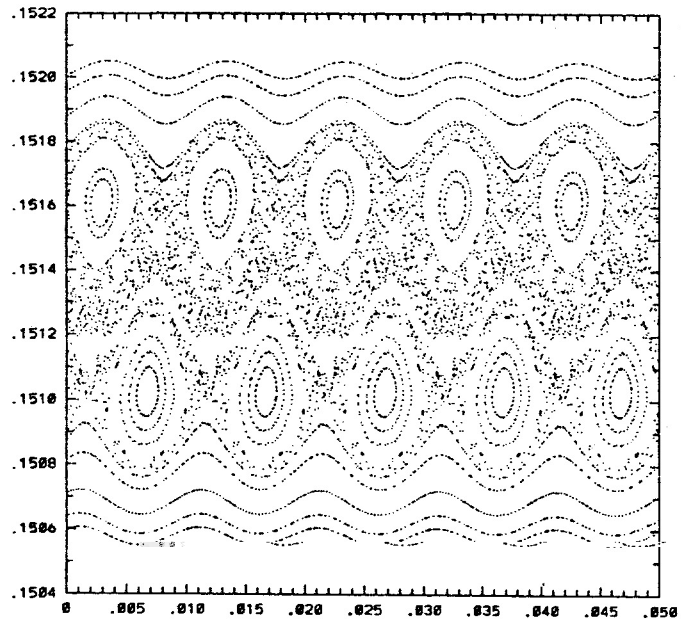


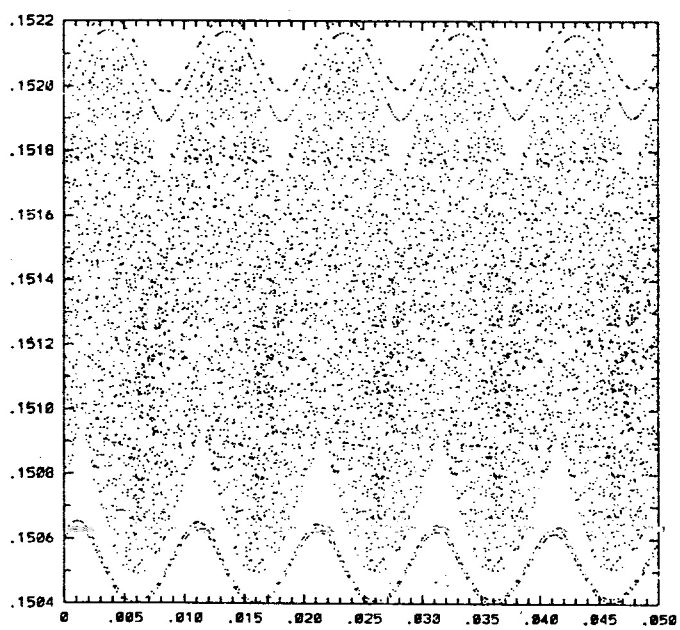
- An island chain of stable and unstable fixed points

  Resonance location: $\omega - \mathbf{n} \cdot \mathbf{\Omega} = 0$

  Resonance width: $\Delta J \sim 2\sqrt{A_n / D_n}$

Motion in More Than One Mode





- Resonance overlap $\Rightarrow$ **large-scale chaotic motion**

Low-Frequency Relativistic Kinetic Eqs

- Once the PSL for single-particle motion is obtained the kinetic equation for collisionless plasma follows very simply...

- For example:
  - Start with the PSL for full-particle motion
  - Obtain general Poisson brackets from the PSL
  - Obtain $\{f, H\}$
  - Liouville's theorem $\Rightarrow df/dt = 0$
  - $0 = df/dt = \{f, H\} + \partial f/\partial t \Rightarrow$ the Vlasov equation

- Since the full-particle to guiding-center transformation is just a coordinate transformation $df/dt = 0$ still holds.

  Just use the Poisson brackets of the transformed PSL to obtain the corresponding (drift-kinetic/gyrokinetic) kinetic equation

- This procedure is much (much!) simpler than other methods of obtaining nonlinear low-frequency kinetic equations.

- Recently, *Chen* [1998] used these methods to derive nonrelativistic quasilinear equations

  The quasilinear equations have the Fokker-Planck form, with a diffusion term and a drag term due to low-frequency wave-particle interactions

- *Brizard and Chan* [1999] have derived a relativistic nonlinear gyrokinetic equation.

  A first step toward a first-principles derivation of a Fokker-Planck radiation belt transport equation

Summary

- The phase-space Lagrangian methods provide ***powerful tools*** for deriving guiding center equations

  The resulting equations

  - Preserve the Hamiltonian properties
    (Energy and phase-space volume conservation)
  - Are easily extended to more general situations
    (E.g., relativistic eqs, wave-particle interactions, self-consistent eqs, ...)

- The generic picture of resonance islands is very useful for analyzing wave-particle interactions.

Concluding Remarks

- **Radiation belt physics is undergoing a renaissance:**
  - **New physical mechanisms**
  - **New theoretical methods**
  - **Increased modeling capability**
  - **Observations from improved instruments**
  - **Multi-spacecraft studies**

- **The coupling of recent theory, modeling, and observations shows great promise for construction of a GGCM radiation belt module**

- **However, much work remains...**
  - **How are the storm-time MeV electrons produced?**
  - **Multi-spacecraft plots of $\overline{f}(\mu, K, L^*, t)$?**
  - **Improved magnetospheric field models are needed.**
  - **Initial and boundary fluxes are needed.**

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