Electromagnetic Coupling of the Magnetosphere to the Ionosphere and Atmosphere

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- Many models of M-I coupling assume height-integrated ionospheric conductivity and an electrostatic ionosphere, so electric field and field-aligned current related by $j_{\|} \sin i = -\nabla \cdot \overleftrightarrow{\Sigma} \cdot \nabla \Phi$
  - Here conductivity can be dynamic, controlled by solar illumination and particle precipitation
  - Thermospheric dynamics is usually ignored
- However, the electrostatic approximation fails since the Hall conductivity couples shear Alfvén waves to compressional fast mode waves with finite $\partial B_{\|}/\partial t$: the inductive ionosphere
- Height-integrated assumption fails for fast time variations and small-scales.
- Coupling to atmospheric fields is also often ignored.

Outline

- The electrostatic ionosphere
  - Reflection of Alfvén Waves
  - Parallel electric fields and auroral scale size
  - Precipitation and Feedback
- The inductive ionosphere
  - Mode coupling
  - Excitation of Pc1 waveguide
- The height-resolved ionosphere
  - Skin depth effects
  - Implications for feedback
- Coupling to the atmosphere
  - Ground signatures
  - Excitation of Schumann resonances

Electrostatic M-I Boundary Condition

- Most magnetospheric modeling treats ionospheric boundary as a conducting slab, with height-integrated conductivities:

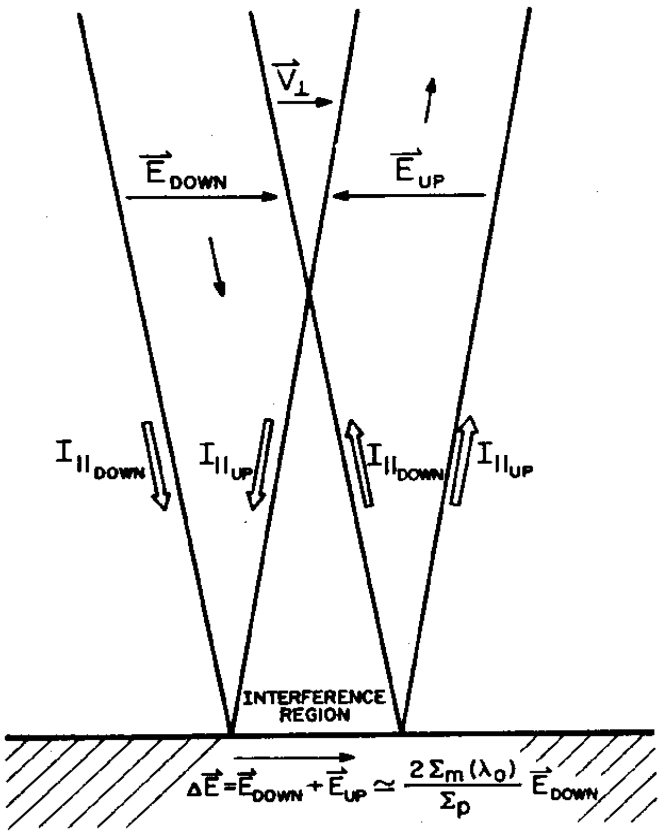
$$\mathbf{I}_{\perp} = \Sigma_P \mathbf{E}_{\perp} + \Sigma_H \hat{\mathbf{b}} \times \mathbf{E}$$

- Boundary condition is based on current continuity and an electrostatic approximation $\mathbf{E} = -\nabla\Phi$ for the electric field:

$$j_{\|} \sin i = -\nabla \cdot \mathbf{I}_{\perp} = \nabla \cdot \left(\overleftrightarrow{\Sigma} \cdot \nabla\Phi \right) = \Sigma_P \nabla^2 \Phi + \nabla\Sigma_P \cdot \nabla\Phi + \nabla\Sigma_H \cdot \left(\hat{\mathbf{b}} \times \nabla\Phi \right)$$

- Conductivity tensor is also modified for non-vertical fields
- Usually current is known, and potential is determined by solving Poisson equation.
- Note that this form implies that only gradients in Hall conductance affect M-I coupling

Reflection of Alfvén Waves by the Ionosphere



(Mallinckrodt and Carlson, 1978)

- Ionosphere acts as terminator for Alfvén transmission line, with admittance $\Sigma_A = 1/\mu_0 V_A$.
- But, impedances don't match: wave is reflected
- Usually $\Sigma_P >> \Sigma_A$, so electric field of reflected wave is tends to cancel incident electric field ("short-circuit")
- For uniform conductances, reflection coefficient is:

$$R = \frac{E_{up}}{E_{down}} = \frac{\Sigma_A - \Sigma_P}{\Sigma_A + \Sigma_P}$$

Parallel Electric Fields: Knight (1973) Relation

- Consider bi-Maxwellian electron population at source region (density n_0, temperatures $T_\|$ and $T_\perp$, magnetic field B_0) in dipole field with upward parallel potential drop Φ. Total current corresponds to those particles that avoid mirroring before reaching the ionosphere. This gives:

$$j_\| = n_0 e \mathrm{v}_{th} \frac{B_I}{B_0}\left[1 - \frac{e^{-xe\Phi/T_\|}}{1+x}\right]$$

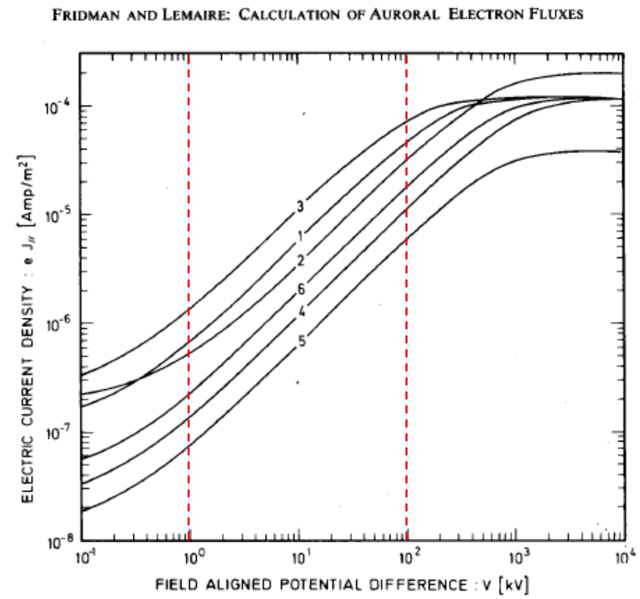
$$x = \frac{T_\| / T_\perp}{B_I / B_0 - 1}$$

$$\mathrm{v}_{th} = \sqrt{T_\| / 2\pi m_e}$$

- Relation is linear for moderate Φ (~1-100 kV)

$$j_{\|,lin} \approx ne\mathrm{v}_{th}\frac{e\Phi}{T_\perp} \equiv K\Phi$$

- For large potential drops, a saturation current is reached: $j_{\|,sat} = nev_{th}B_I/B_0$
- Important point: Knight relation only gives the field-aligned current resulting from an assumed potential drop. It does NOT explain the existence of parallel electric fields.



Role of the Ionosphere: Electrostatic Scale Size

(Lyons, 1980)

- Ionosphere closes field-aligned currents: $j_{\|} = -\nabla_{\perp} \cdot \left(\overleftrightarrow{\Sigma} \cdot \mathbf{E} \right)$
- For electrostatic conditions, uniform ionosphere, only Pedersen conductivity matters: $j_{\|} = \Sigma_P \nabla_{\perp}^2 \Phi_I$
- Assume the linear Knight relation is valid: $j_{\|} = K(\Phi_I - \Phi_0)$
- Combining these leads to equation for potential: $\left(1 - L^2 \nabla_{\perp}^2\right) \Phi_I = \Phi_0$
- Here $L = \sqrt{\Sigma_P / K}$ is electrostatic auroral scale length.
- For Σ_P = 10 mho and $K = 10^{-9}$ mho/m^2, L = 100 km
- Parallel potential drops only exist on scales shorter than L

Effects of $E_{||}$ on Alfven Wave Reflection: Alfvenic Scale Size

- If assume linear Knight relation $j = K\Phi$, Alfven wave reflection is modified (Vogt and Haerendel, 1998)
- Reflection coefficient same $R = (\Sigma_A - \Sigma_{eff})/(\Sigma_A + \Sigma_{eff})$ if replace Pedersen conductivity with effective conductivity

$$\Sigma_{eff} = \frac{\Sigma_P}{1 + k_\perp^2 L^2}$$

where $L = \sqrt{\Sigma_P / K}$

- This leads to a new scale where the Alfvén wave is absorbed (providing energy to auroral particle acceleration) given by

$$L_A = \sqrt{\Sigma_A / K} \sim 10 \text{ km}$$

Bi-modal distribution of auroral arc widths

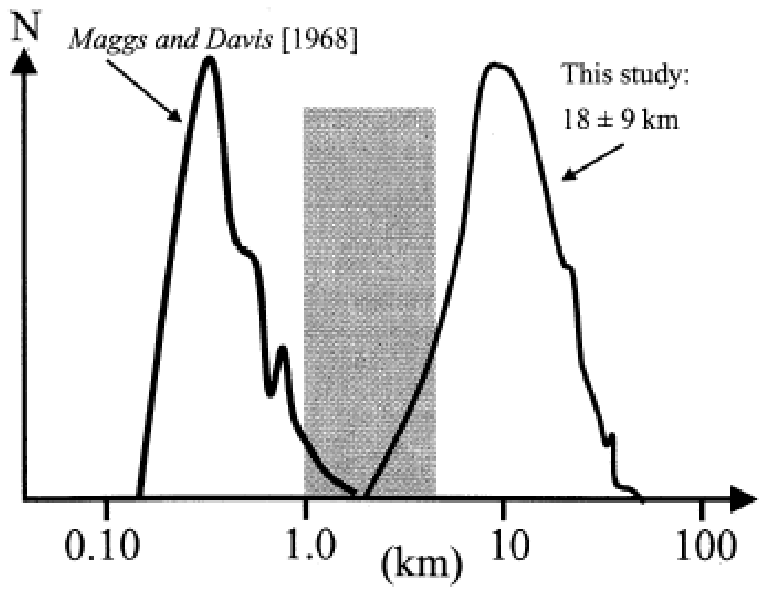


Figure 4. Combining width distributions taken with narrow-field and all-sky lenses indicates a gap in arc width near 1 km. The gray area represents a region of uncertainty in resolving 1-km-scale structures, pointing to a need for observations optimized for this scale.

(Knudsen et al., *Geophys. Res. Lett., 28,* 705, 2001)

Auroral arcs show a bi-modal distribution, with a peak at very small scales of < 1 km and a second peak at about 10 km. Larger-scale structures are consistent with linear calculations; however, narrow-scale arcs are still not understood.

Conductance Variations from Precipitation

- Ionospheric conductances result from Solar EUV, discrete electron precipitation and diffuse precipitation, balanced by recombination
  - Solar contribution parameterized by solar zenith angle (Vickrey+,1981):

$$\Sigma_P = 5 \text{ mho} \sqrt{\cos\chi} \qquad \Sigma_H \approx 2\Sigma_P$$

  - Electron precipitation parameterized by Robinson (1987) relations:

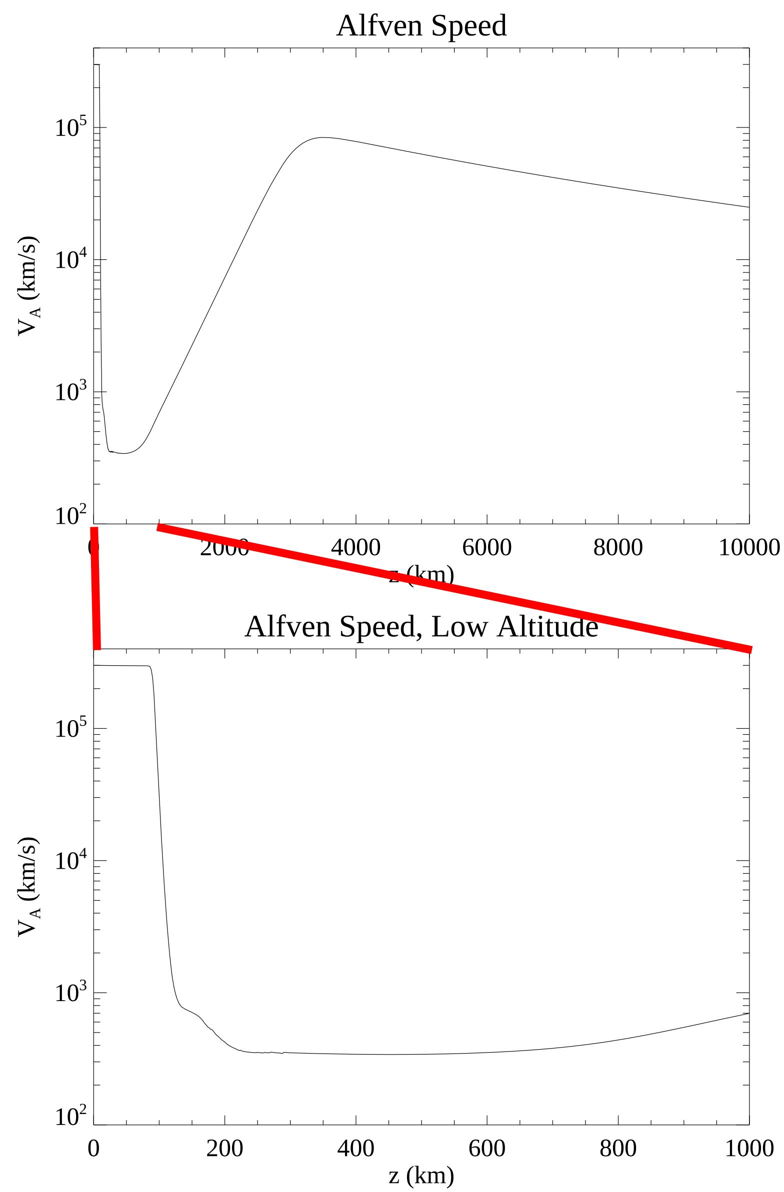
$$\Sigma_P(\text{mho}) = \frac{40E_0}{16+E_0^2} F_E^{1/2} \qquad \Sigma_H / \Sigma_P = 0.45 E_0^{0.85}$$

(E_0 in keV; F_E in mW/m^2)

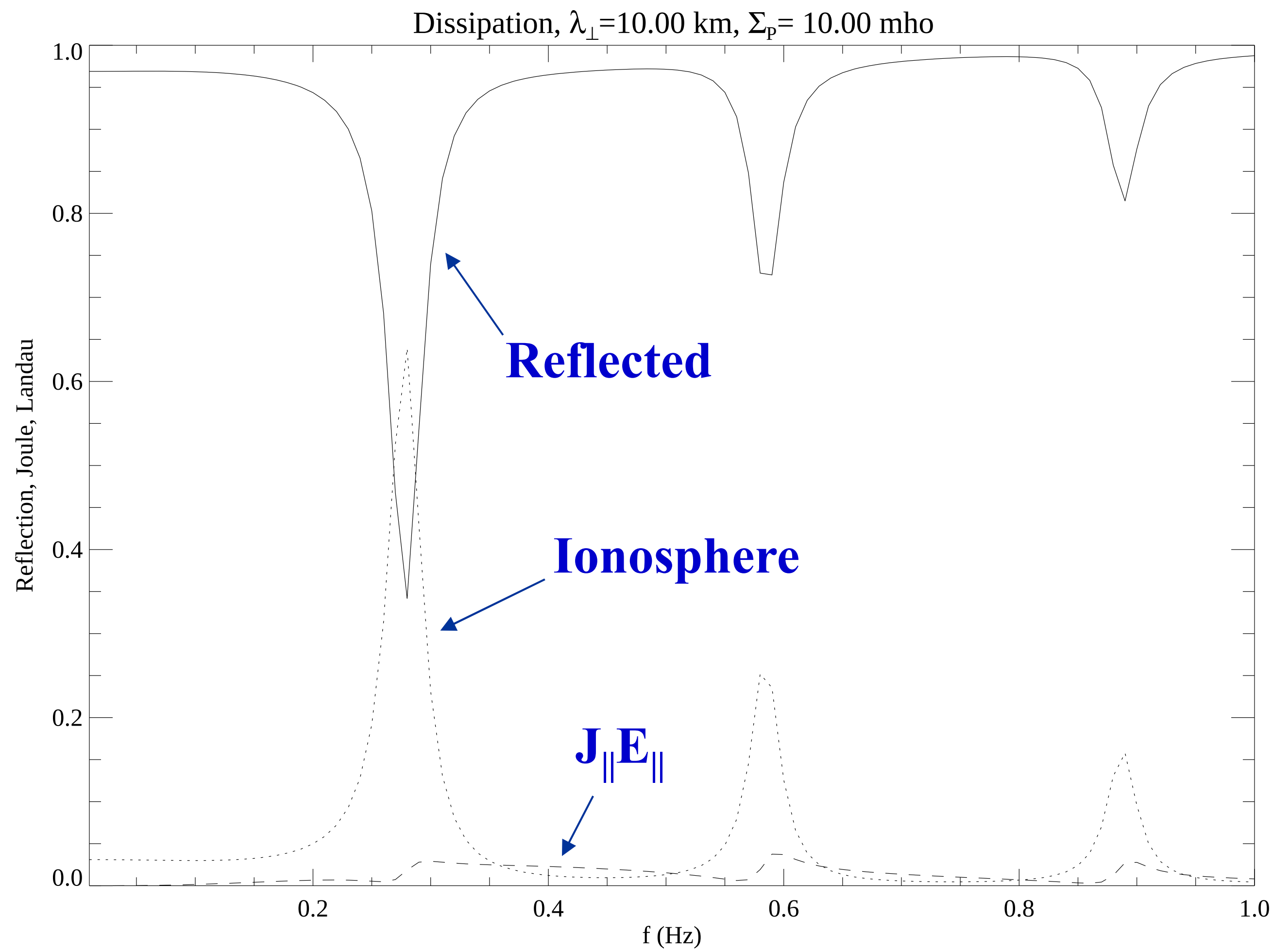
  - For diffuse precipitation, Maxwellian assumed; for discrete, Knight relation used: $E_0 = e\Phi_{||}$, $F_E = j_{||}\Phi_{||}$
  - Rule of thumb: one electron-ion pair produced for every 35 eV of precipitating electron energy (Rees, 1963)
- More recently, conductances from Coupled Thermosphere-Ionosphere Model (e.g., Raeder, 2001, 2008).

Frequency Dependence: Ionospheric Alfvén Resonator

- Many M-I coupling models also assume a constant Alfvén speed in the magnetosphere
- However, the Alfvén speed rises sharply above ionosphere due to exponential fall of plasma density.
- Wave propagation speed goes back to the speed of light at altitudes below the ionosphere.
- The minimum in Alfvén speed in ionosphere forms a resonant cavity for shear Alfvén waves (Ionospheric Alfvén Resonator) and a waveguide for fast mode waves in 1-10 s period range.



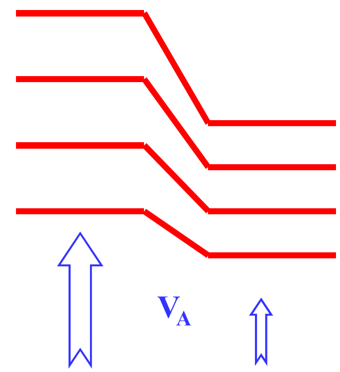
Frequency dependence of reflection in IAR

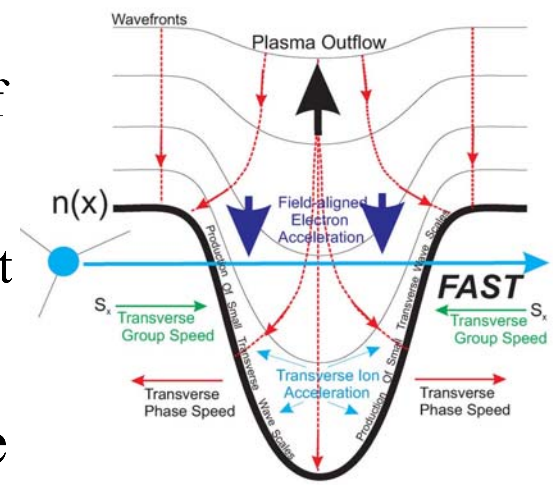


Plot shows reflected energy flux, ionospheric dissipation and dissipation due to the parallel electric field in inertial Alfvén wave in the ionospheric Alfvén resonator

Small Spatial Scales: Phase Mixing

- Gradients in the Alfvén speed lead to phase mixing, producing smaller perpendicular scales (basic mechanism behind field line resonance.)
- Such gradients are always present, especially in boundary regions:
  - Plasma Sheet Boundary Layer: poleward boundary of aurora
  - Boundaries of aurora density cavities (e.g., Chaston et al., 2006, at right)
- Scale length estimated to be ~ $(\Sigma_A/\Sigma_P)\ L_0$, where $\Sigma_A = 1/\mu_0 V_A$ is Alfvén conductance and L_0 is gradient scale length.

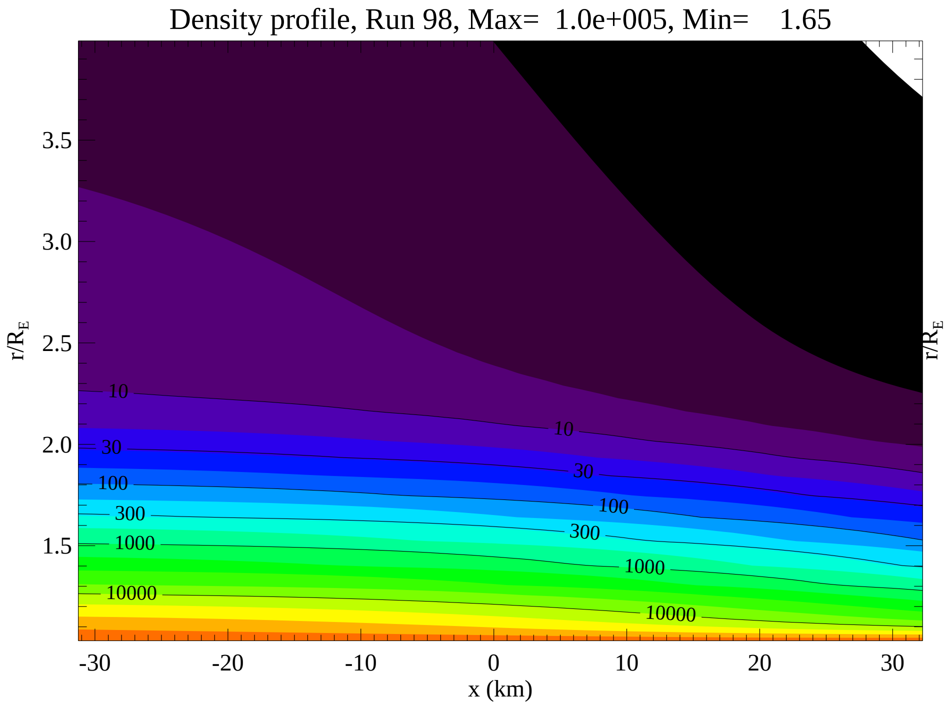




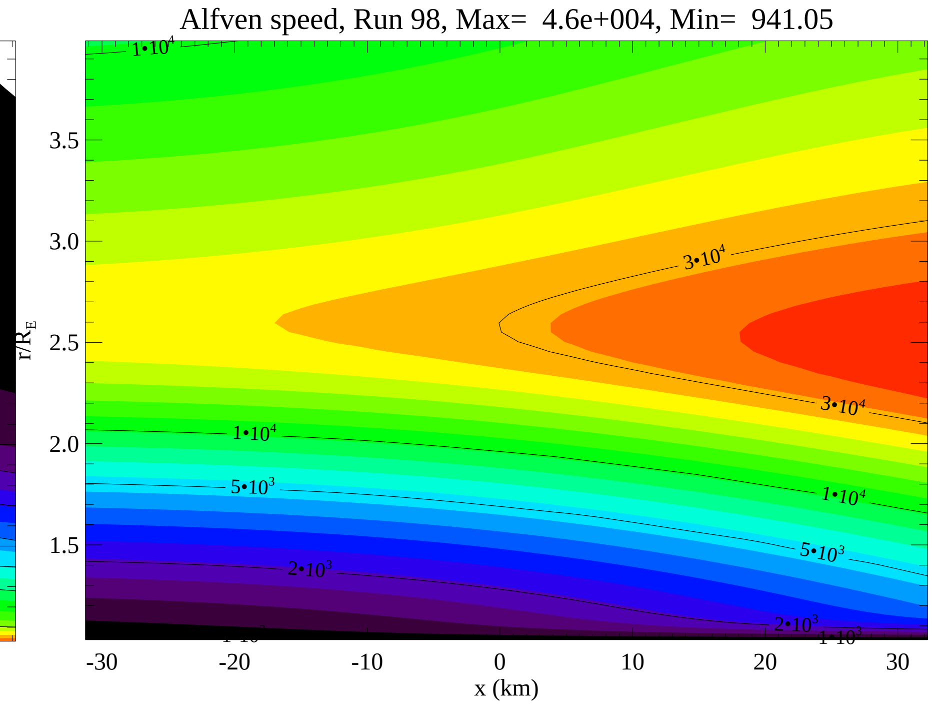
Simulation of Phase Mixing in IAR

- Simulations of linear wave propagation including electron inertia effect were made in a overall perpendicular density gradient.

Density

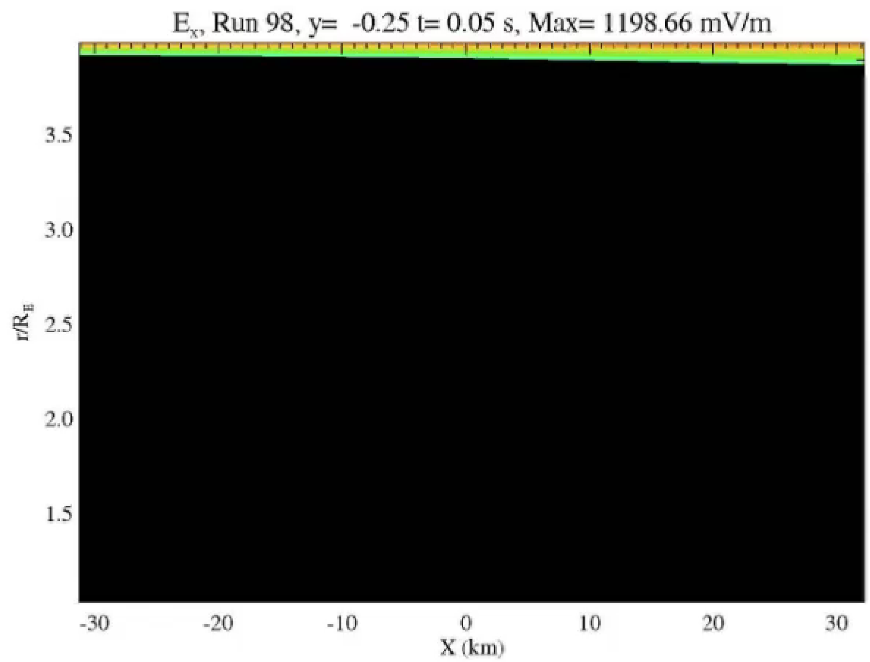


Alfvén speed

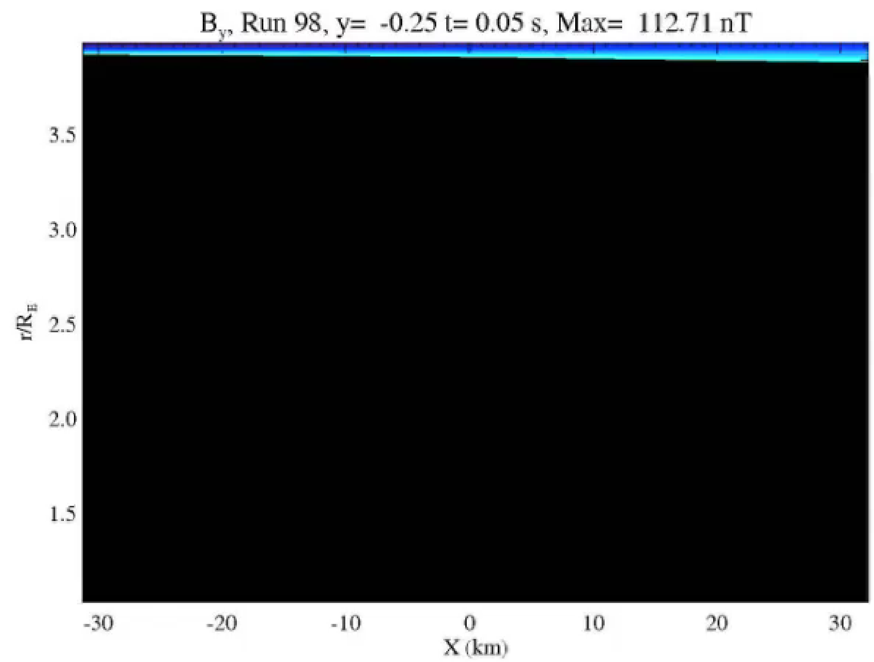


Simulation results

E_x



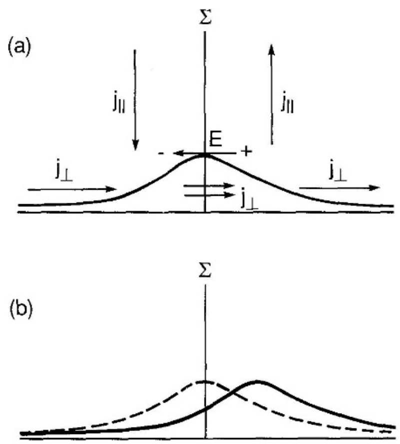
B_y



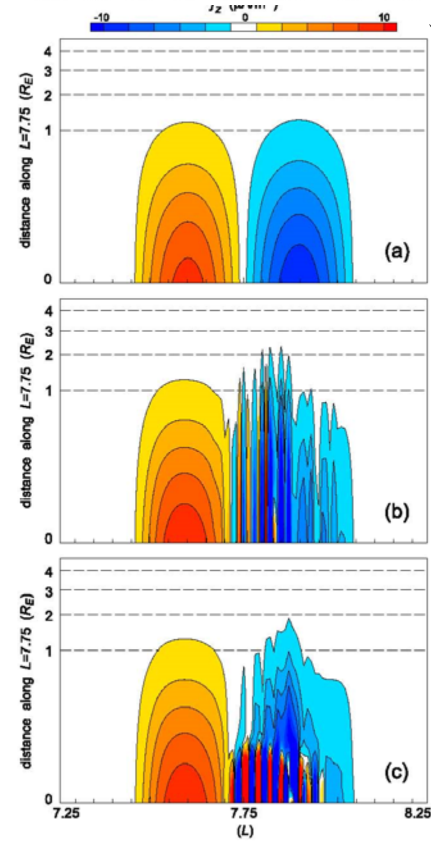
- Simulation initiated with uniform pulse across system oscillating at 1 Hz.
- Interference between up and downgoing waves leads to structuring of fields.
- Series of harmonics seen due to change of IAR eigenfrequencies.
- Waves phase mix to ~ 1 km scale waves.

Ionospheric Feedback

- In presence of background convection, fluctuations in conductivity can give rise to a feedback interaction
  - ➢ Conductivity enhancement requires either polarization electric field or closure of enhanced currents by field-aligned currents
  - ➢ Upward field-aligned current (downward electrons) can enhance conductivity
  - ➢ Reflections in IAR or from conjugate ionosphere can lead to instability
- Small-scale structures can form in large-scale downward current regions (blue and violet in lower figure)
- Scale size limited by parallel resistivity, < 1 km
- However:
  - ➢ Theory has only been developed with height-integrated conductivity (but new results from height-resolved model, later)



(Lysak, 1990)

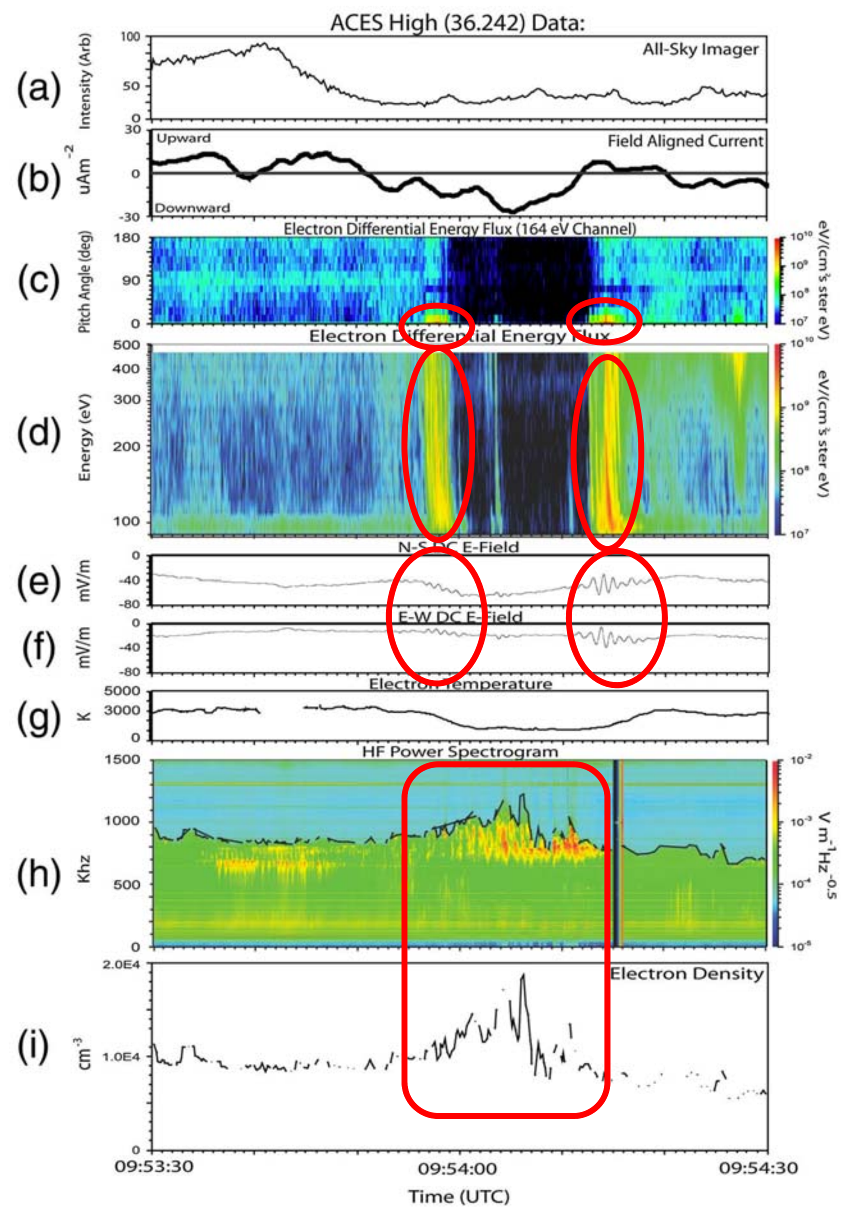


(Streltsov and Lotko, 2008)

Observations of Feedback Interactions

- Cohen et al. (2013) have observed particles and fields suggestive of feedback interaction.
- Strong field-aligned (panel c) electron fluxes with broad energy range (panel d) seen at edges of downward current region (panel b)
- Electron precipitation associated with low frequency (~ 1 Hz, panels e and f) electric fluctuations
- Enhanced density in downward current region (panels h and i): signatures of upwelling from below?
- Consistent with modeling of feedback instability, but also possible phase mixing at density gradients.

COHEN ET AL.: ACES OBSERVATIONS OF IFI



Inductive Ionospheric Boundary Condition
(Yoshikawa and Itonaga, 1996; Lysak and Song, 2006)

- Electrostatic boundary condition only deals with the shear Alfven mode that carries field-aligned current; it does not provide a boundary condition for the fast mode waves.
- A more general boundary condition can be found by integrating Ampere's Law over the ionosphere:

$$\mu_0 \overleftrightarrow{\Sigma} \cdot \mathbf{E} = \hat{\mathbf{r}} \times \Delta \mathbf{B}$$

$\Delta\mathbf{B}$ is difference between magnetic fields above and below ionosphere

- For vertical field lines and uniform $\overleftrightarrow{\Sigma}$, taking the divergence yields the usual electrostatic condition, while taking the curl gives a second condition:

$$\Sigma_P \nabla_\perp \cdot \mathbf{E}_\perp - \Sigma_H \hat{\mathbf{r}} \cdot \left(\nabla \times \mathbf{E}_\perp\right) = -j_\parallel$$

$$\Sigma_H \nabla_\perp \cdot \mathbf{E}_\perp + \Sigma_P \hat{\mathbf{r}} \cdot \left(\nabla \times \mathbf{E}_\perp\right) = (1/\mu_0)\Delta\left(\partial B_r / \partial r\right)$$

- These equations illustrate the coupling of the shear mode (div E) and the fast mode (curl E) by the Hall conductivity.
- Inductive condition straightforward to implement with finite magnetic zenith angle at all latitudes.
- Note that this equation requires knowledge of **B** in the atmosphere.

The Atmospheric Solution

- Implementation of this model requires a solution below the ionosphere.
- Assume atmosphere is perfectly insulating, ground is perfectly conducting: magnetostatic approximation
- Then in atmosphere can use magnetic scalar potential

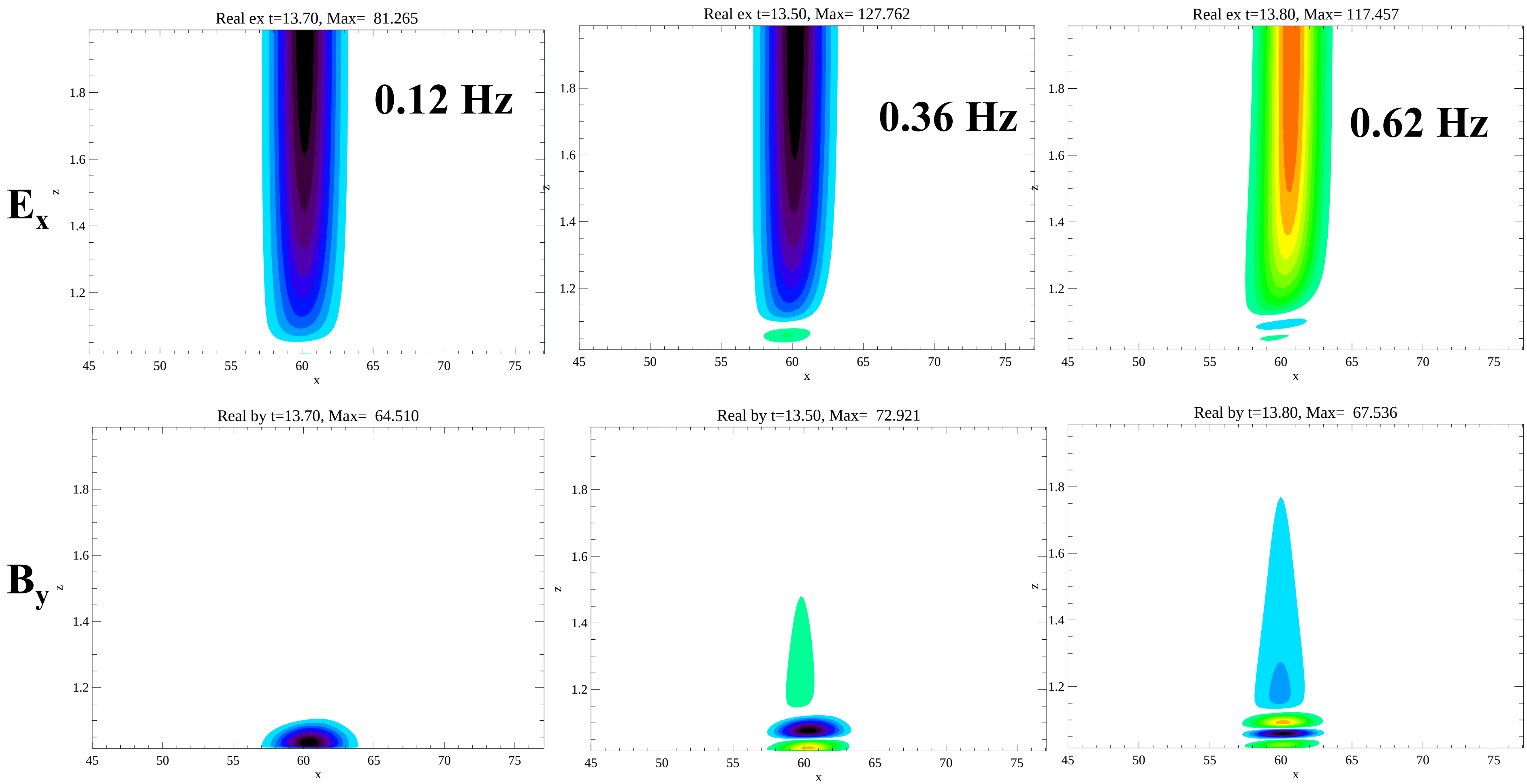
$$\nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{B} = 0 \ \Rightarrow \mathbf{B} = \nabla \Psi,\ \ \nabla^2 \Psi = 0$$

- Field is "frozen-in" to ground, so $B_r = \partial \Psi / \partial r = 0$
- Radial magnetic field is continuous through layer, so Ψ is set by matching solution to simulation B_r
- Solution can be written in terms of spherical harmonics, modified to fit simulation boundaries:

$$\Psi(r,\theta,\varphi) = \sum_{l.m} \left(A_{lm} r^{\nu_l} + B_{lm} r^{-(\nu_l+1)} \right) Y_{lm}(\theta,\varphi)$$

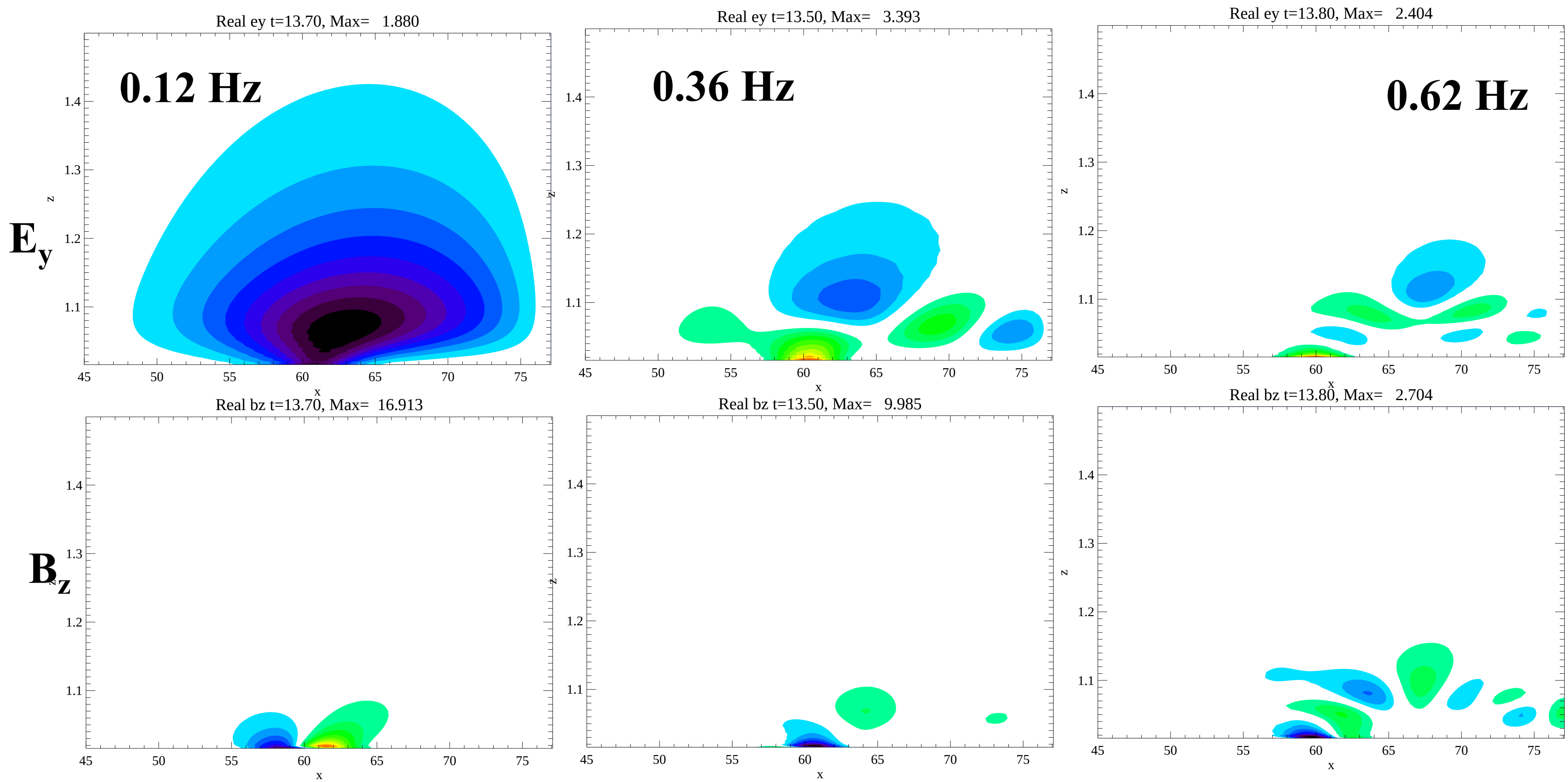
- Note that this solution allows direct calculation of ground magnetic fields as well as field just below ionosphere.

Simulation of IAR with Inductive Ionosphere



- E_x (top) and B_y (bottom) mode structures for 0.12, 0.36, and 0.62 Hz runs showing harmonic structures in IAR. Only region below 2 R_E is shown

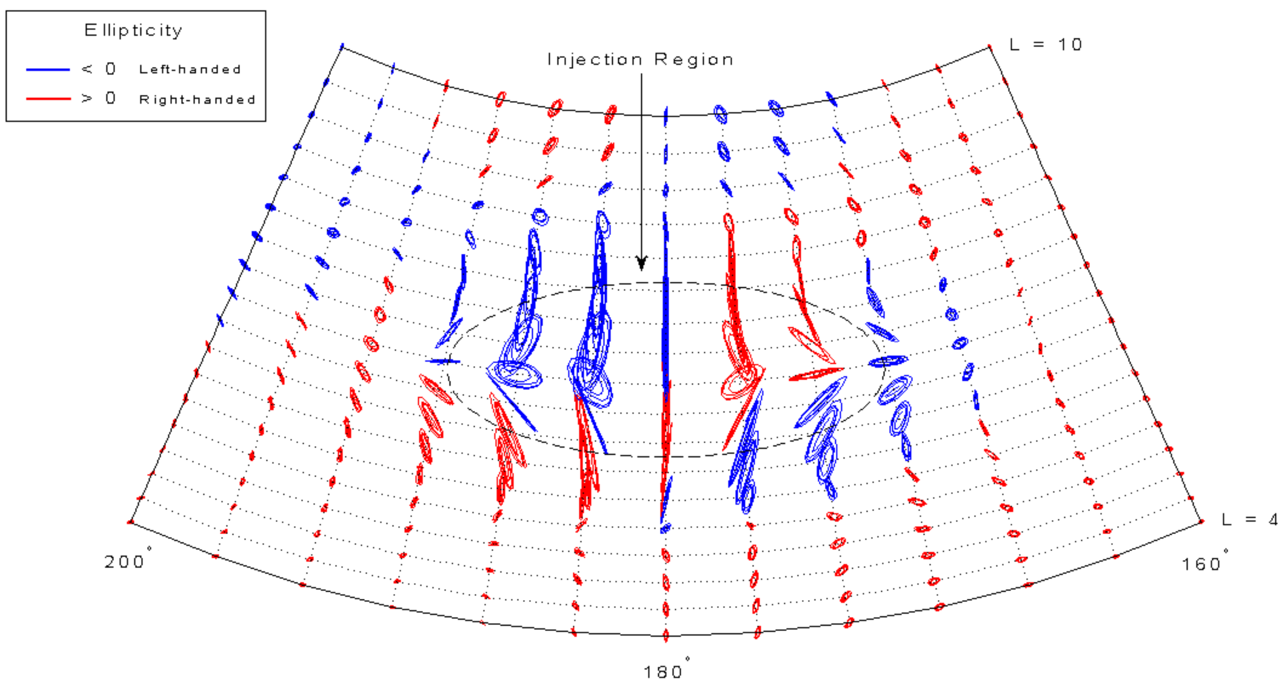
Mode Coupling: Effect of Hall conductivity



- Hall currents couple modes, as seen in E_y (top) and B_z (bottom) components. Fast mode is evanescent for fundamental (left) but higher harmonics propagate. Only region below 1.5 R_E shown.

Mode Coupling: Effect of Hall conductivity

- Hall currents couple shear mode and fast mode: Fast mode propagates horizontally in Pc1 waveguide (e.g., Fraser, 1976; Engebretson et al., 2002)
- This propagation gives characteristic pattern of polarization, reproduced in simulations of Woodroffe and Lysak (2012):

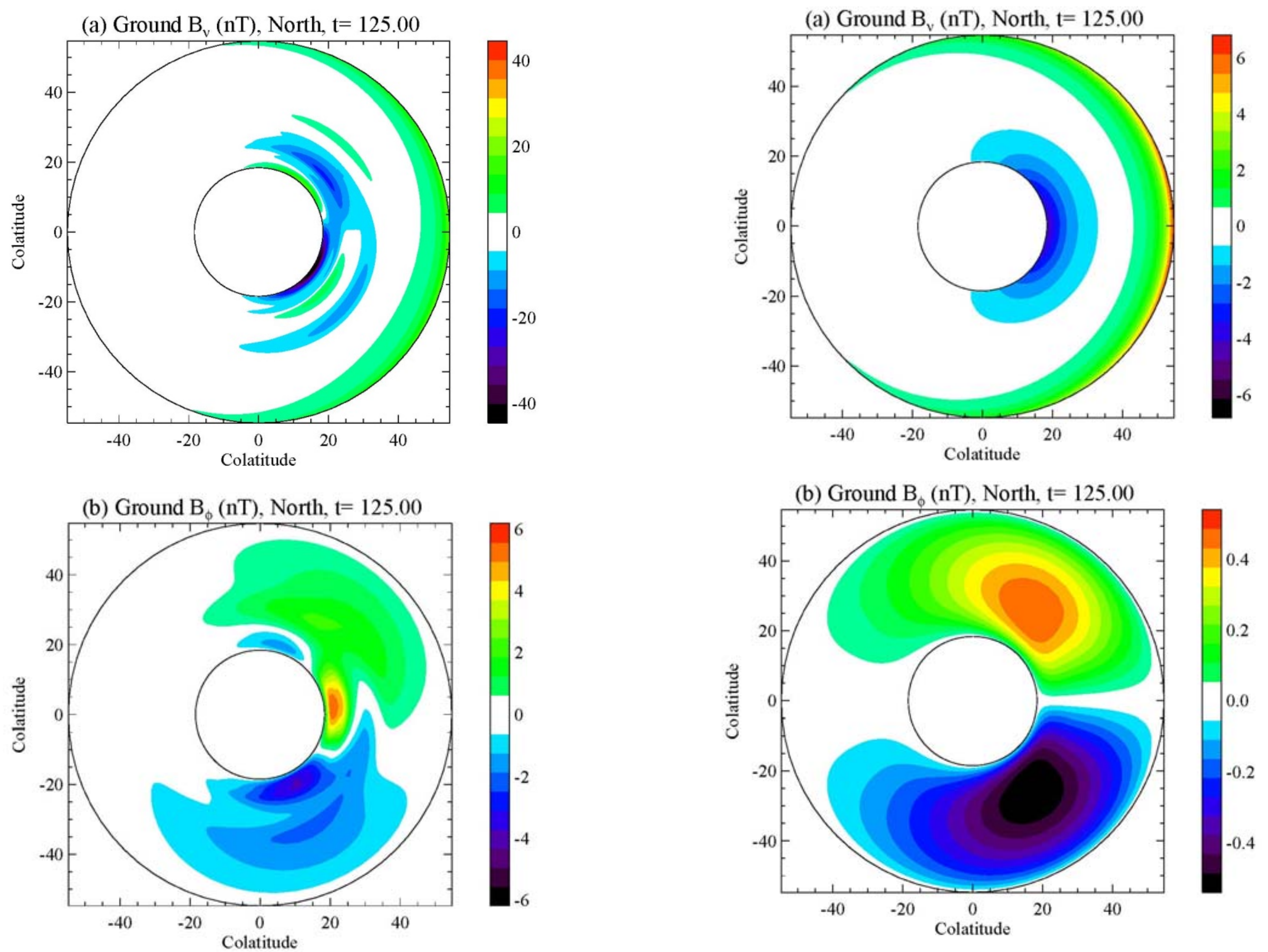


Effect of Hall conductivity

- Even uniform Hall conductivity breaks dawn-dusk symmetry in ground fields, in contrast to electrostatic model

Northward field

Eastward field

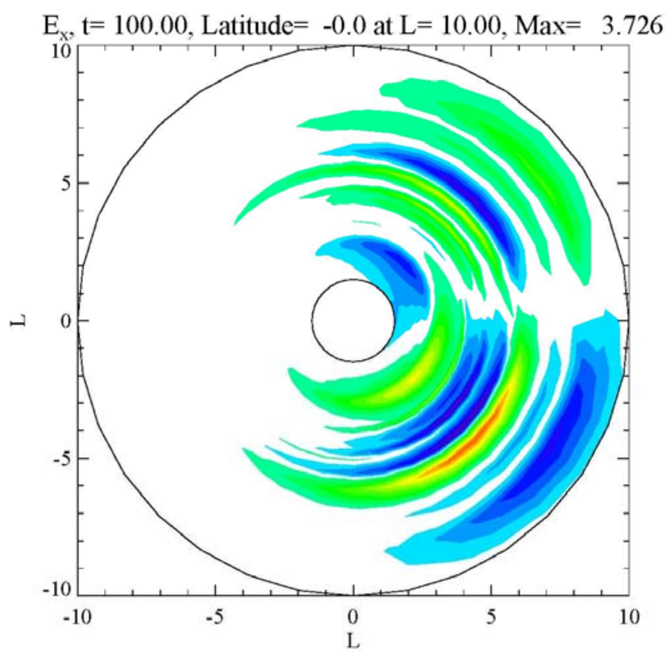


$\Sigma_P = 1.8$ mho, $\Sigma_H = 3.1$ mho

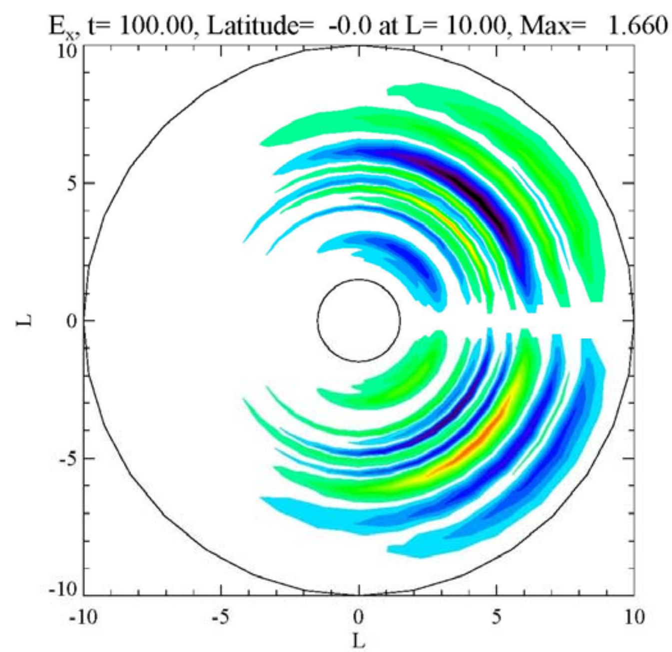
$\Sigma_P = 1.8$ mho, $\Sigma_H = 0$ mho

Effect of Hall conductivity

- Hall conductivity breaks dawn-dusk symmetry in convection
- Radial electric field (positive outward) plotted in equatorial plane



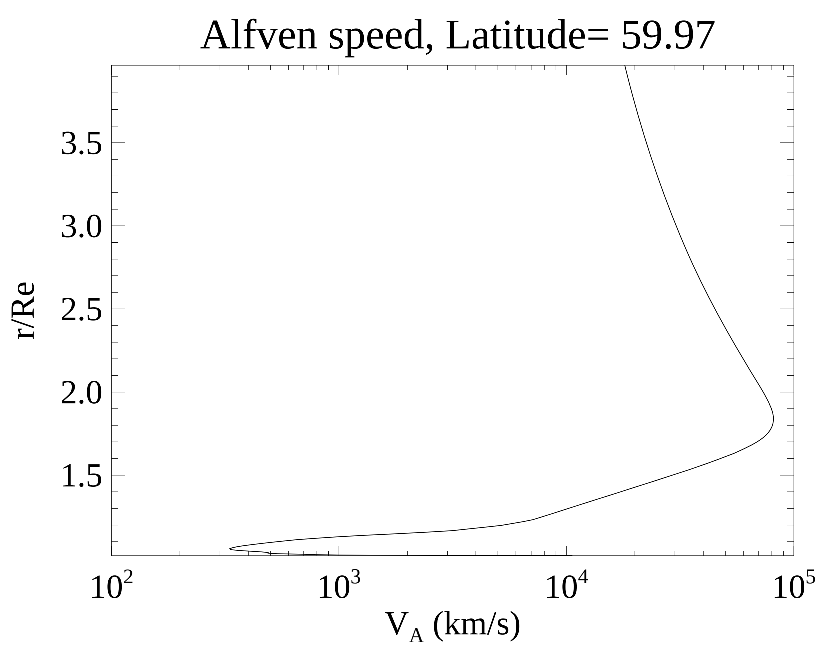
Σ_P = 1.8 mho, Σ_H = 3.1 mho

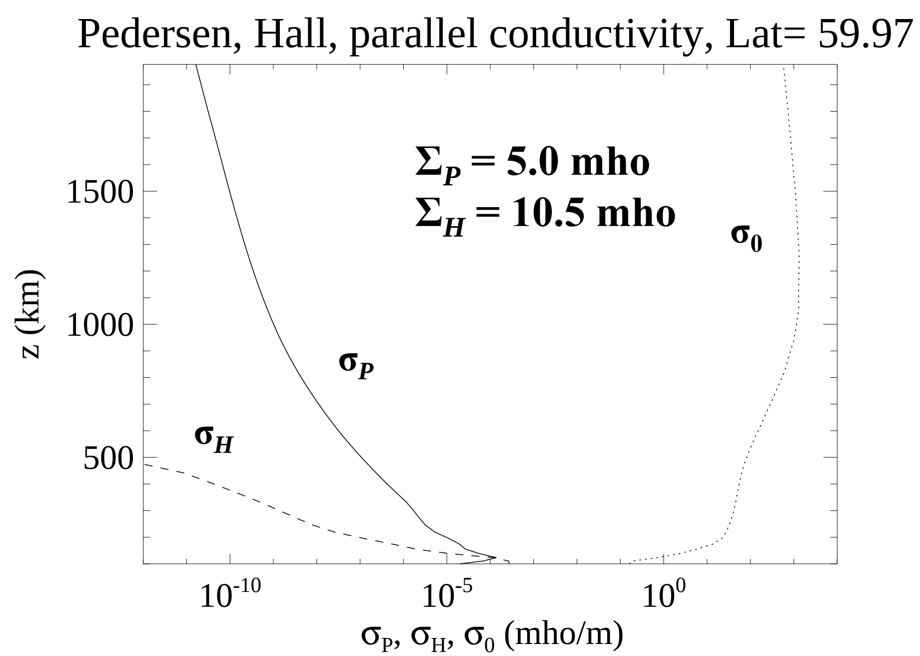


Σ_P = 1.8 mho, Σ_H = 0 mho

Ionosphere is not a height-integrated sheet!

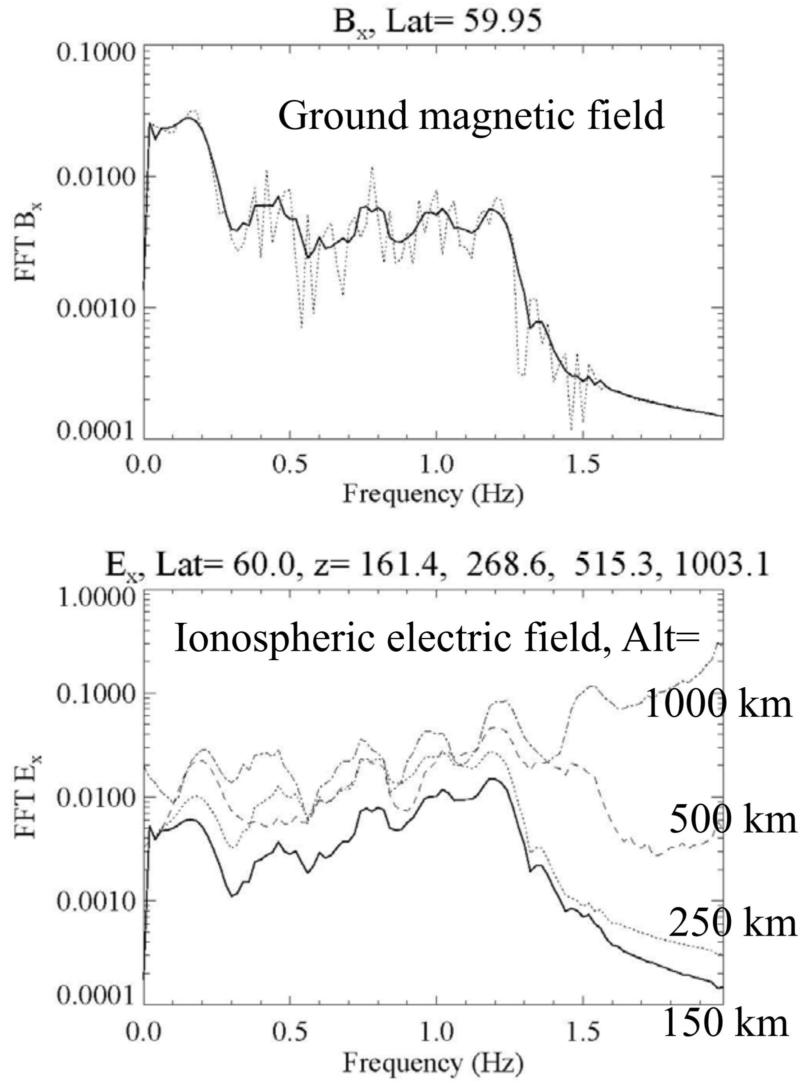
- Sheet ionosphere is not a good approximation for higher frequencies and short perpendicular wavelengths
- Need to include distributed Pedersen, Hall and parallel conductivities
- Note that parallel conductivity can limit propagation of short wavelength waves (e.g., *Forget et al.*, 1991; *Lessard and Knutsen*, 2001)
  - Scale length $L_{res} = \lambda_e \sqrt{\nu_e / \omega}$ ~150 m in ionosphere (*Lysak and Song*, 2002)
- Example: Daytime, Solar Min parameters in *Kelley* (1989)





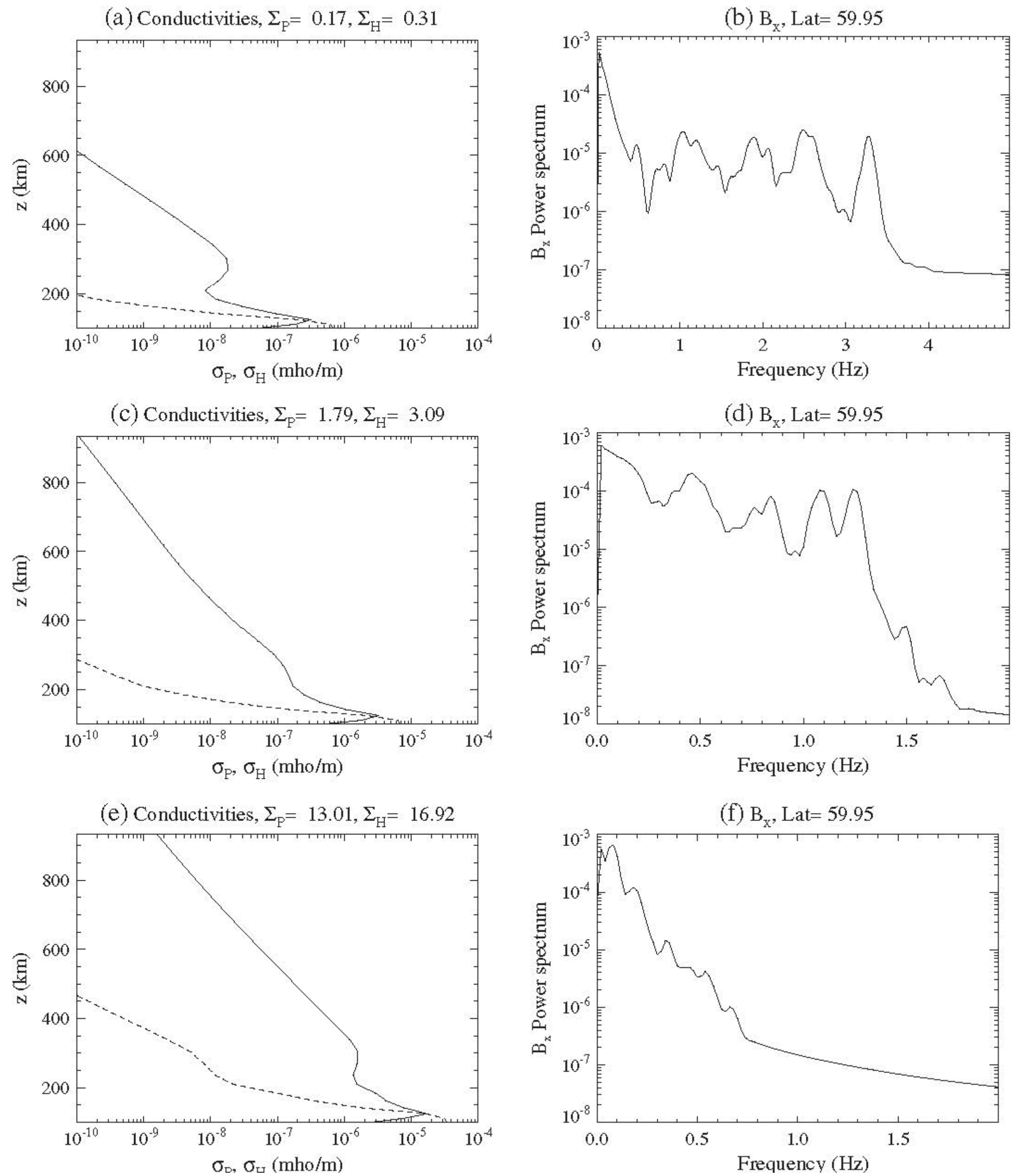
Ionospheric Shielding Effect

- The sheet ionosphere assumption also fails at higher frequencies.
- Ionospheric Pedersen conductivity acts to shield higher frequency waves (collisional skin depth) $\delta = \sqrt{2/\mu_0 \omega \sigma_P}$
- Results are shown from a numerical model of Alfvén wave propagation including height-resolved ionosphere (*Lysak et al.*, 2013)
- Model is driven with a broad-band "white noise" spectrum consisting of 100 waves from 0-2 Hz with equal amplitudes and random phases.
- It can be seen that the higher frequency components are attenuated at lower altitudes in the ionosphere



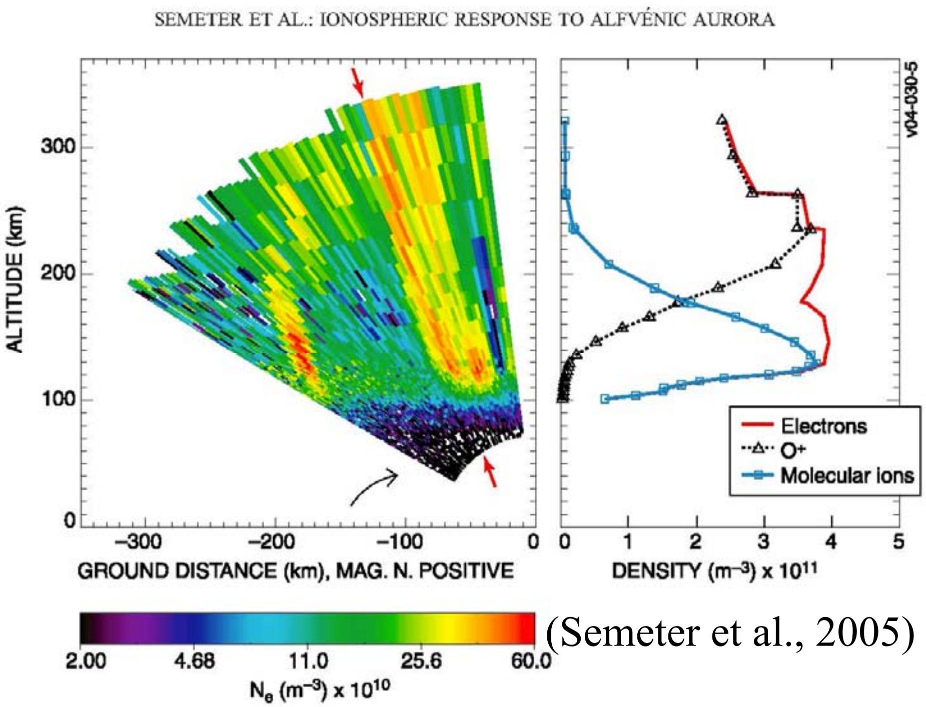
Variation of Conductivity

- Ground magnetic field plotted for runs with different conductivity profiles (*Lysak et al.*, 2013).
  - Solid lines Pedersen conductivity, dashed line Hall conductivity
- Cutoff frequency for ground fields lower for higher conductivity profile.



Horizontal Gradients

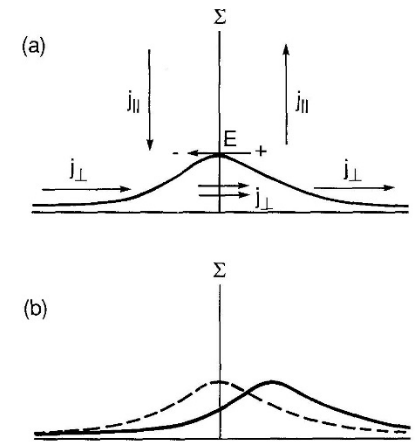
- Ionosphere is not only vertically stratified (as assumed in most M-I coupling models), but can have perpendicular gradients
  - Especially true in auroral ionosphere where localized electron precipitation can give columns of ionization
- Gradients in Alfvén speed can give rise to phase mixing and/or feedback interactions, producing smaller-scale, intense field-aligned currents
- In presence of background convection, conductivity gradients can also lead to strong currents



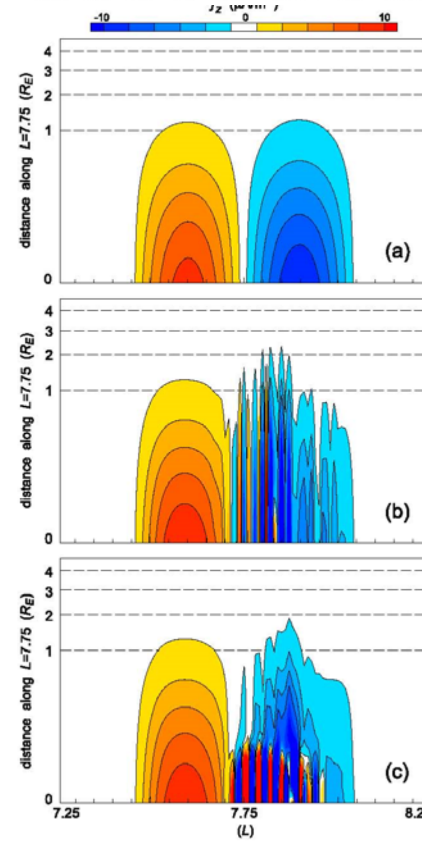
(Semeter et al., 2005)

Ionospheric Feedback

- In presence of background convection, fluctuations in conductivity can give rise to a feedback interaction
  - ➢ Conductivity enhancement requires either polarization electric field or closure of enhanced currents by field-aligned currents
  - ➢ Upward field-aligned current (downward electrons) can enhance conductivity
  - ➢ Reflections in IAR or from conjugate ionosphere can lead to instability
- Small-scale structures can form in large-scale downward current regions (blue and violet in lower figure)
- Scale size limited by parallel resistivity, < 1 km
- However:
  - ➢ Theory has only been developed with height-integrated conductivity



(Lysak, 1990)

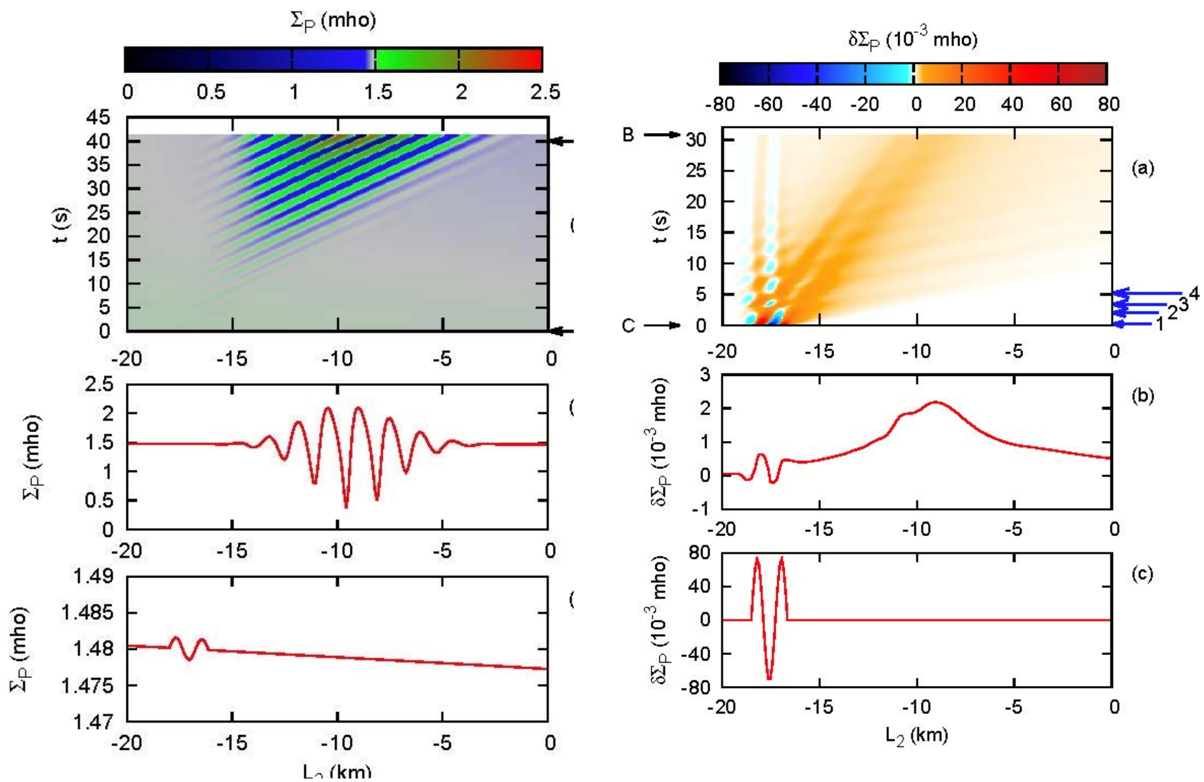


(Streltsov and Lotko, 2008)

Effect of Resolved Ionosphere on Feedback

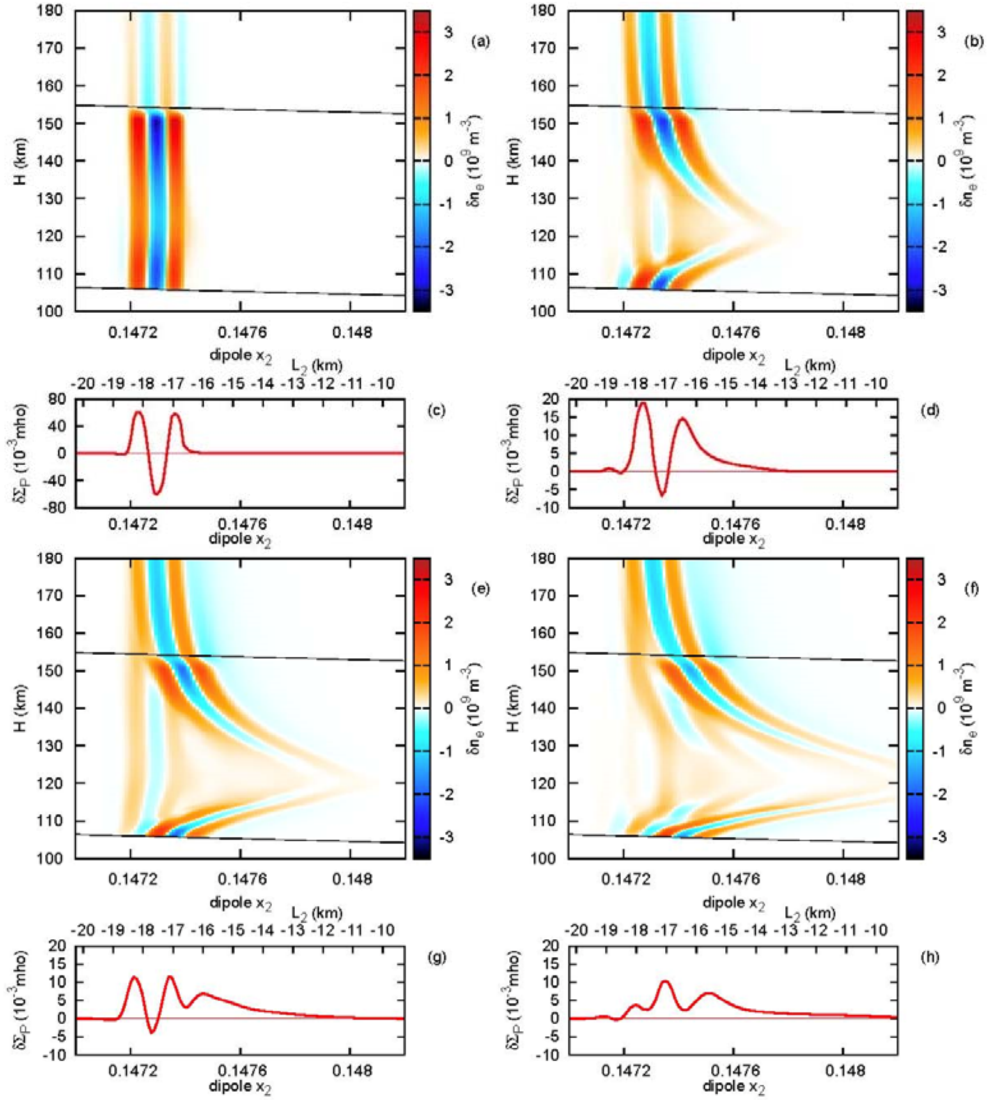
- Sydorenko and Rankin (Fall mini-GEM) presented results suggesting that resolved ionosphere suppresses feedback instability
- Simulation at left uses height-integrated model, at right is height-resolved
- Panel (a) is space-time plot, (b) is at 40 s, (c) is initial condition
- Conductivity is slightly enhanced in resolved case, but doesn’t produce small-scale structure.
- Note: this model only includes electron flow, not additional ionization due to precipitation

(Figure courtesy Sydorenko and Rankin)



Why is feedback suppressed?

- Feedback instability depends on presence of large-scale convection electric field.
- Strong gradient in Pedersen conductivity shears ion flow in this field, preventing formation of narrow structures



(Figure courtesy Sydorenko and Rankin)

Coupling to Atmosphere: Earth-Ionosphere Waveguide

- Is magnetostatic model for coupling to ground correct?
- Kikuchi and co-workers (1978…2014) have proposed excitation of the TM mode in the Earth-ionosphere waveguide
- Magnetostatic model is low-frequency limit of evanescent TE mode, valid if thickness of atmosphere $\ll c/\omega$, i.e., if frequency $\ll$ 1 kHz
- However, TM mode can propagate at lower frequencies, with cutoff at lowest Schumann resonance at ~ 8 Hz: ***not*** at zero frequency as is often stated.
- When system is driven at ULF frequencies, atmosphere is more of a resonant cavity than a waveguide.

Observations of Schumann Resonances

(Jackson, Chapter 8)

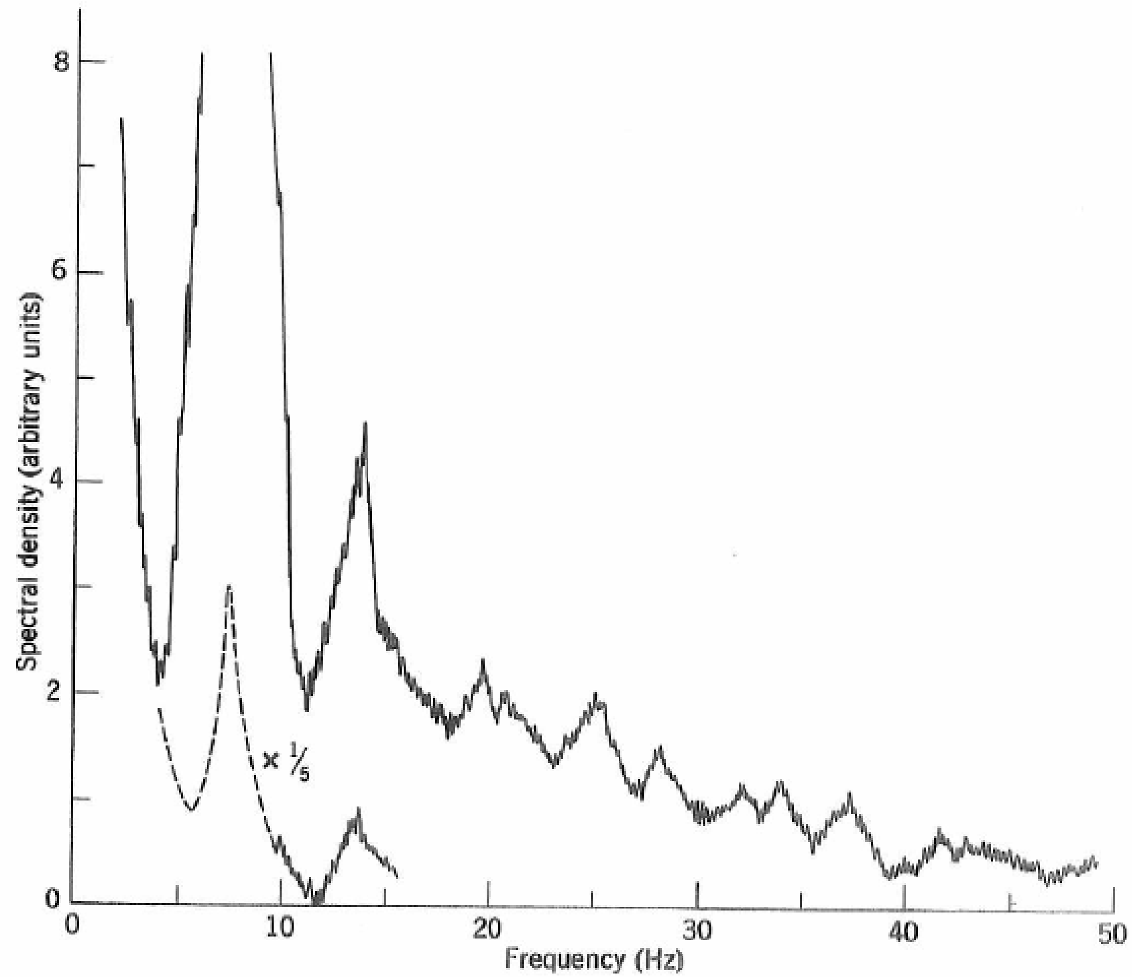


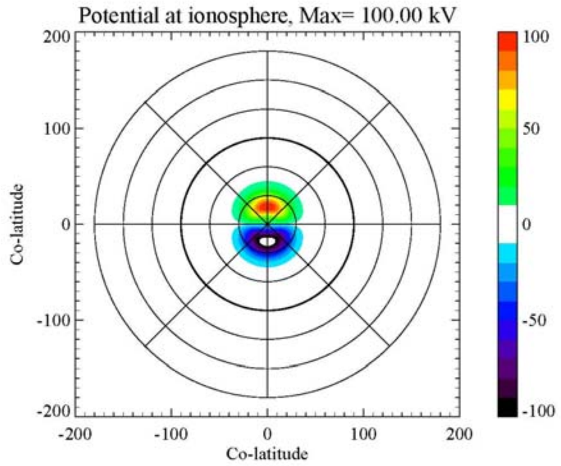
Figure 8.9 Typical noise power spectrum at low frequencies (integrated over 30 s), observed at Lavangsdalen, Norway on June 19, 1965. The prominent Schumann resonances at 8, 14, 20, and 26 Hz, plus peaks at 32, 37, and 43 Hz as well as smaller structure are visible. [After A. Egeland and T. R. Larsen, *Phys. Norv.* 2, 85 (1967).]

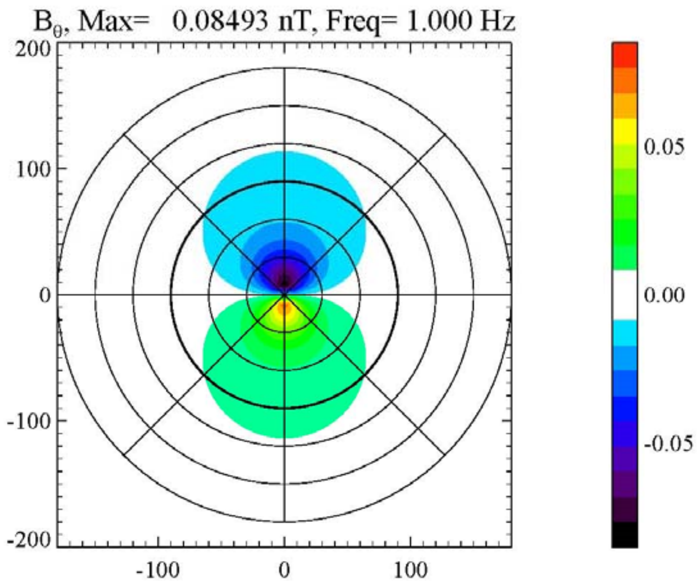
Excitation of the TM mode

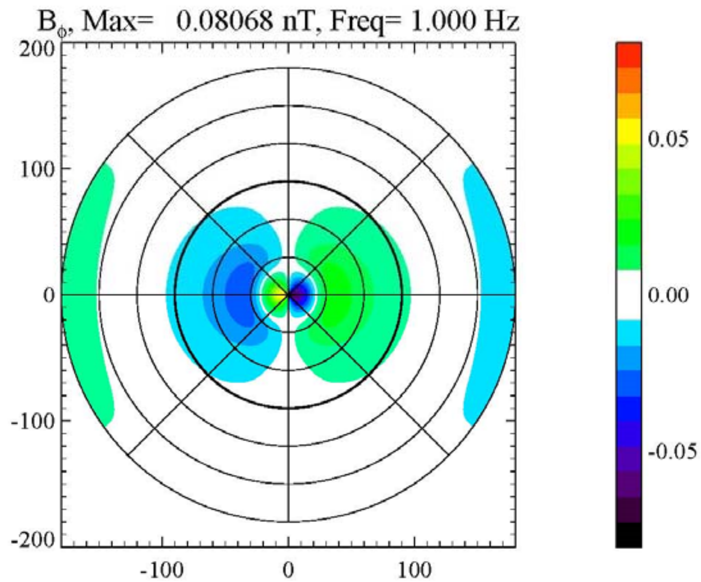
- By definition, TM mode has $B_r = 0$
- Thus, curl of horizontal electric field is zero, so it can be represented by a potential.
- Consider the atmosphere to be a spherical shell with conducting boundaries.
- Then, by Maxwell's equations, the horizontal electric field is continuous across the atmosphere-ionosphere boundary
- So, consider a two-cell like convection pattern oscillating at ULF frequency
- Solution can be written as spherical harmonic expansion; radial dependence can be written in terms of spherical Bessel functions

Model Results

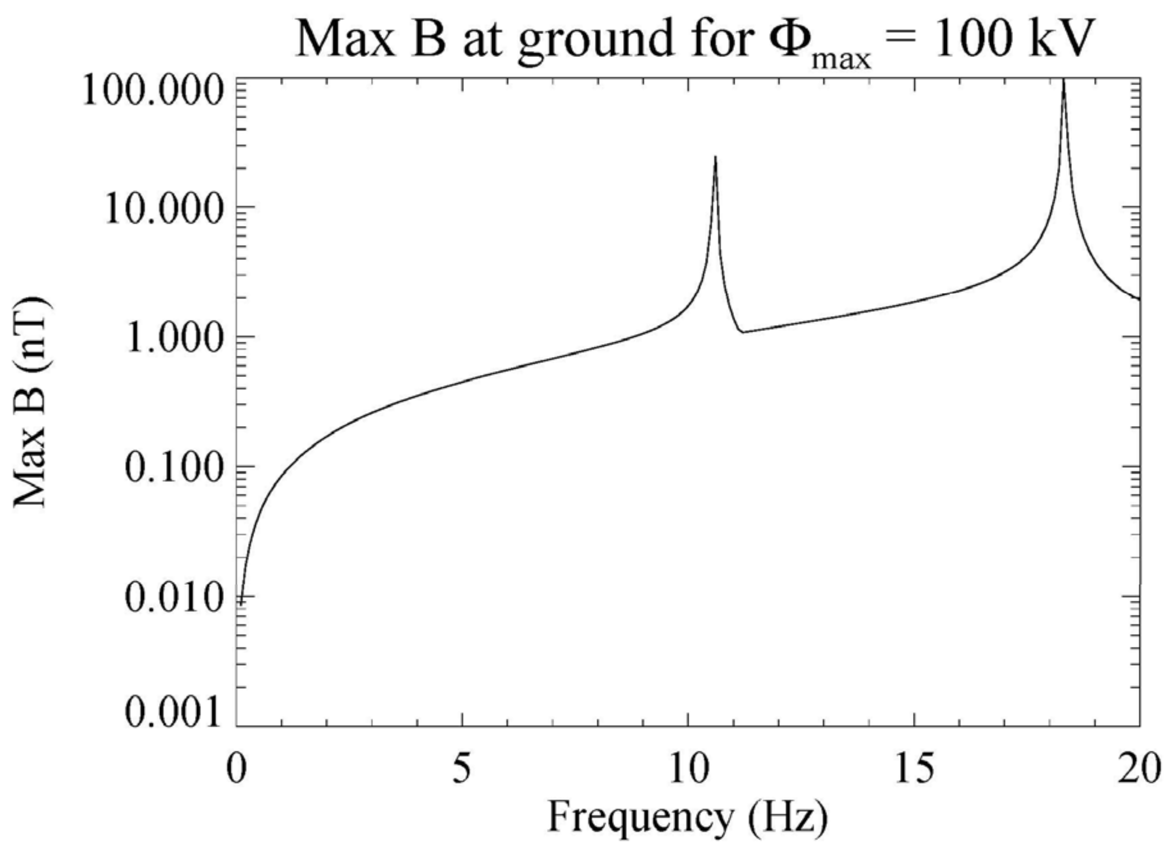
- A 100-kV two-cell potential pattern oscillating at 1 Hz is imposed
  - This corresponds to max electric field of 78 mV/m
  - Circular contours every 30° of latitude; equator shown by thicker line
- Southward (B_θ) and Eastward (B_φ) ground fields shown; Fields < 0.1 nT found:





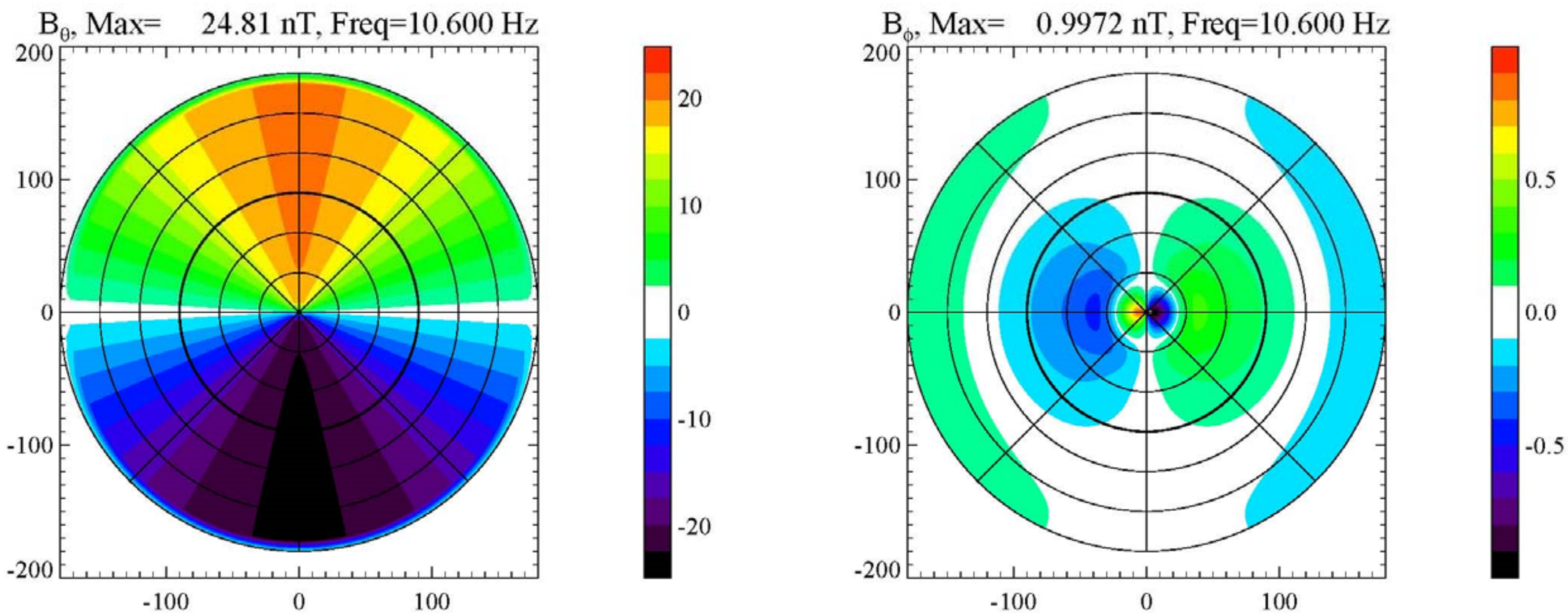


Frequency Dependence



- Maximum field anywhere on ground as function of frequency
- Fields less than 1 nT except near Schumann resonances (10.6 Hz in this model)
- TM fields generally much less than TE fields seen previously
- Equatorial fields even smaller than max.

Fields at Schumann Resonance

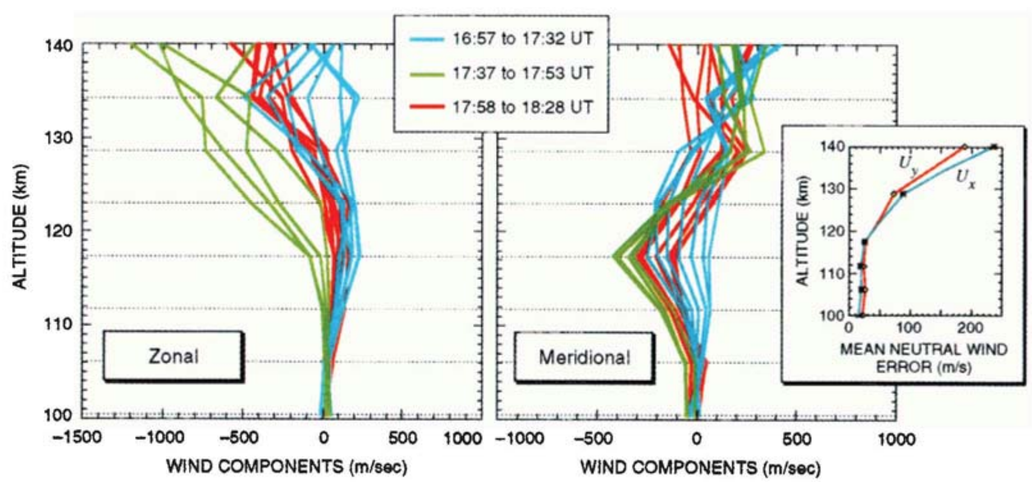


- At Schumann resonance, meridional magnetic field is uniform across latitude

Not to mention neutral winds…

- And most magnetospheric physicists would prefer not to!
- Or at least assume a single neutral wind rest frame in which Ohm's Law is valid, $\mathbf{j} = \vec{\boldsymbol{\sigma}} \cdot (\mathbf{E} + \mathbf{V}_n \times \mathbf{B})$
- However, Thayer (1998) derived neutral winds from radar data (assuming steady-state ion motion) and showed large vertical gradients in wind speed.

THAYER: HEIGHT-RESOLVED JOULE HEATING RATES AND NEUTRAL WINDS



So what needs to be done?

- Progress in understanding the details of the magnetosphere-ionosphere interaction requires a good knowledge of the state of the ionosphere
  - Time resolutions ~ 1 sec to resolve IAR frequencies
  - Spatial resolution ~ 1 km to resolve auroral arc scales
- A system like EISCAT-3D would be a great advance in this regard
- Is it possible to characterize neutral atmosphere at similar time and space scales?
  - Or is it necessary given large inertia in thermosphere; how fast can thermospheric winds vary?

Mathematical Details

(adapted from Jackson, section 9.7)

- Take spherical harmonic analysis of imposed potential:

$$\Phi\left(R_I,\theta,\varphi\right)=\sum_{l,m}\Phi_{lm}Y_{lm}\left(\theta,\varphi\right)\ \text{with}\quad \Phi_{lm}=\int d\Omega\,\Phi(R_I,\theta,\varphi)Y_{lm}^*(\theta,\varphi)$$

- Then potential at all r becomes

$$\Phi(r)=\sum_{l,m}\Phi_{lm}\frac{F_l(kr)G_l(kR_E)-F_l(kR_E)G_l(kr)}{F_l\left(kR_I\right)G_l(kR_E)-F_l\left(kR_E\right)G_l(kR_I)}Y_{lm}\left(\theta,\varphi\right)$$

- Here $F_l\left(x\right)=\dfrac{d}{dx}\left(xj_l\left(x\right)\right)$, $G_l\left(x\right)=\dfrac{d}{dx}\left(xy_l\left(x\right)\right)$ and j_l and y_l are spherical Bessel functions. Also $k = \omega/c$ and R_E and R_I are radii of Earth and ionosphere.

- Then horizontal electric and magnetic fields are

$$\mathbf{E}_\perp=-\nabla_\perp\Phi=-\Phi_{lm}\frac{G_l\left(kR_E\right)F_l\left(kr\right)-F_l\left(kR_E\right)G_l\left(kr\right)}{F_l\left(kR_I\right)G_l\left(kR_E\right)-F_l\left(kR_E\right)G_l\left(kR_I\right)}\left(\hat{\theta}\frac{1}{r}\frac{\partial Y_{lm}}{\partial\theta}+\hat{\varphi}\frac{im}{r\sin\theta}Y_{lm}\right)$$

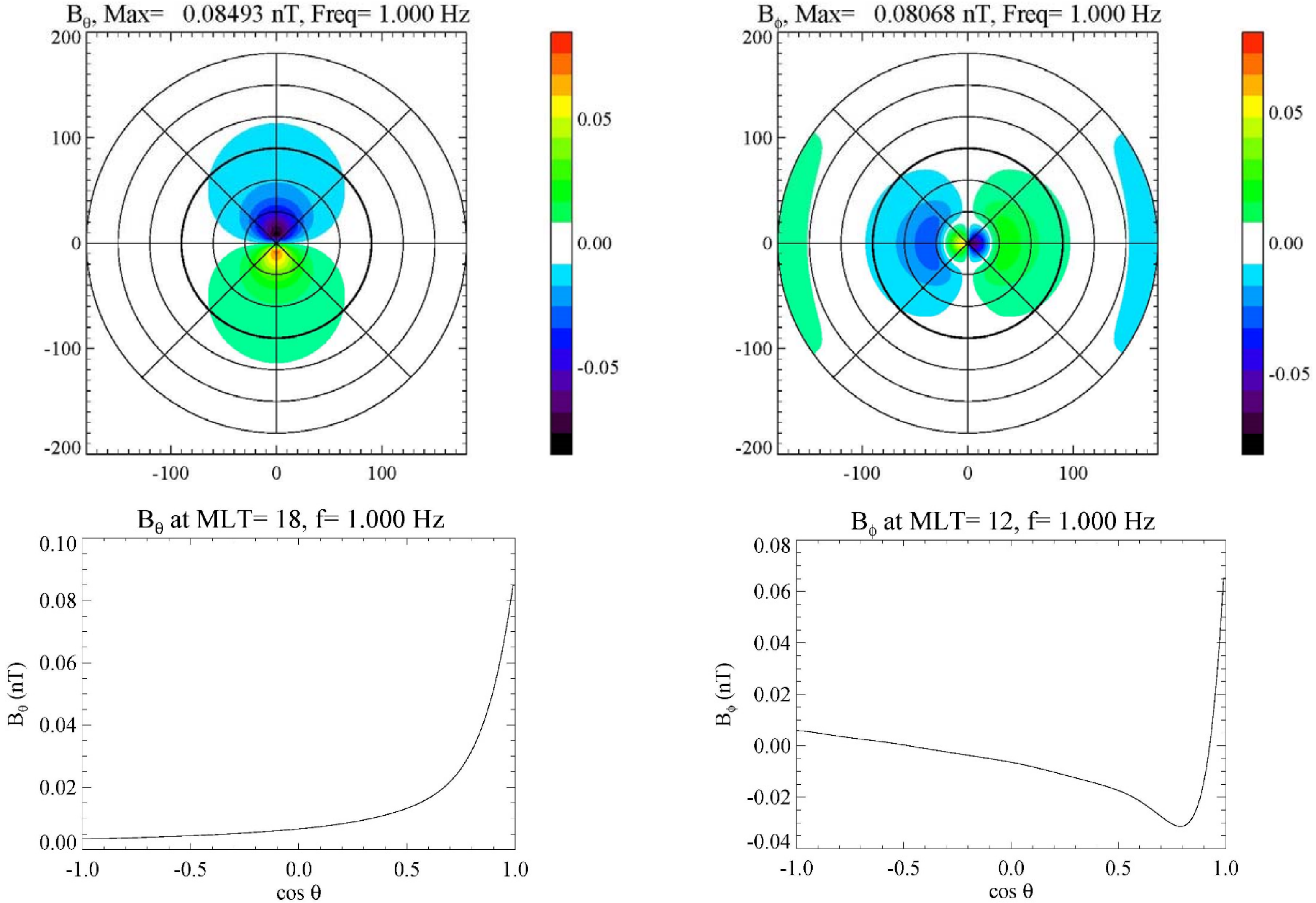
$$\mathbf{B}_\perp=\frac{ik}{c}\Phi_{lm}\frac{G_l\left(kR_E\right)j_l\left(kr\right)-F_l\left(kR_E\right)y_l\left(kr\right)}{F_l\left(kR_I\right)G_l\left(kR_E\right)-F_l\left(kR_E\right)G_l\left(kR_I\right)}\left(\hat{\theta}\frac{im}{\sin\theta}Y_{lm}-\hat{\varphi}\frac{\partial Y_{lm}}{\partial\theta}\right)$$

- In thin-shell approximation, denominator proportional to $l(l+1)-(kR_E)^2$. giving normal modes at $f=(c/2\pi R_E)\sqrt{l\left(l+1\right)}=10.6\text{ Hz}$

- Too high by 30%: shells are not perfect conductors (as noted by Jackson)

Model Results

- Contours of magnetic field, along with line plots at 18 MLT (B_θ) and 12 MLT (B_φ)



Effective Pedersen Conductance

- Parallel electric fields (Lysak, 1998):
  - Assume linear Knight relation, $j_{\|} = K\Phi$, where Φ is parallel potential drop
  - Defines a characteristic scale length, $L = \sqrt{\Sigma_P / K}$
  - Then effective Pedersen conductance becomes

$$\Sigma_{P,eff} = \frac{\Sigma_P}{1 + k_\perp^2 L^2}$$

- Inductive ionosphere (Yoshikawa and Itonaga, 1996; Lysak, 2001)
  - Let d be height of ionosphere, and assume $k_\perp V_A >> \omega$
  - Then effective conductance becomes

$$\Sigma_{P,eff} = \Sigma_P + \frac{\Sigma_H^2}{\Sigma_P + i(k_\perp / \mu_0 \omega)\left(1 + \coth k_\perp d\right)}$$

Solution on a sphere (cont.)

- Now general solution for scalar potential is given by:

$$\Psi(r,\theta,\varphi) = \sum_{l.m} \left(A_{lm} r^{\nu_l} + B_{lm} r^{-(\nu_l+1)} \right) y_{lm}(\theta,\varphi)$$

- At the ground ($r = R_E$), $\partial\Psi/\partial r = 0$, which implies:

$$B_{lm} = \frac{\nu_l}{\nu_l + 1} R_E^{2\nu_l+1} A_{lm}$$

- At the ionosphere ($r = R_I$), the solution matches the magnetospheric solution ($\partial\Psi/\partial r = B_r$), which gives the coefficients A_{lm}:

$$A_{lm} = \frac{1}{\nu_l R_I^{\nu_l-1} \left[1 - \left(R_E / R_I \right)^{2\nu_l+1} \right]} \int d\Omega \; B_r\left(R_I,\theta,\varphi\right) y_{lm}^{*}(\theta,\varphi)$$

- Now the horizontal magnetic fields at ground or ionosphere can be found by the horizontal gradients of Ψ.